Non-ergodicity and growth are compatible for 1-D local interaction

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Abstract

We present results of Monte Carlo simulation and chaos approximation of a class of Markov processes with a countable or continuous set of states. Each of these states can be written as a finite (finite case) or infinite in both directions (infinite case) sequence of *pluses* and *minuses* denoted by $\oplus$ and $\ominus$. As continuous time goes on, our sequence undergoes the following three types of local transformations: The first one, called *flip*, changes any minus into plus and any plus into minus with a rate β. Another, called *annihilation*, eliminates two neighbor components with a rate α whenever they are in differents states. The third one, called *mitosis*, doubles any component with a rate γ. All of them occur at any place of the sequence independently. Our simulations and approximations suggest that with appropriate positive values of α, β and γ this process has the following two properties. *Growth:* In the finite case, as the process goes on, the length of the sequence tends to infinity with a probability which tends to 1 when the length of the initial sequence tends to ∞. *Non-ergodicity:* The infinite process is non-ergodic and the finite process keeps most of the time at two extremes, occasionally swinging from one to the other.

Introduction

Since the first studies of the Ising model, it became common among physicists to recognize the qualitative difference between one-dimensional and multi-dimensional case for all multi-component models with local interaction. This lore crystallized in the shape of the "positive rates conjecture" (see [Liggett, pp. 178, 201]) and was brilliantly refuted by Peter Gács ([Gacs, Gray]). However, the cases, when a random process with one-dimensional local interaction shows some form of non-ergodicity, remain non-trivial and for this reason still attract attention; our task is to provide another case of this sort. Our ultimate goal is to study the case, which we call infinite. In this case configurations are infinite in both directions sequences of *pluses* and *minuses* denoted by $\oplus$ and $\ominus$ respectively. However, this case is not yet defined rigorously. In addition, every computer has a finite memory, so any computer simulation in fact is a simulation of some finite process. For these reasons along with infinite processes, we deal with analogous finite processes, which are easy to define and which we in fact model.

In the finite case, to avoid complications at the ends, we use configurations called "circulars". A circular is just a finite sequence of pluses and minuses, but terms of this sequence, called components, are enumerated by remainders modulo $|C|$ where $|C|$ is the length (that is the number of components) of the circular, rather than natural numbers. (In the literature this is called sometimes periodic condition.) Figure 1 shows a circular C with length $n = |C|$ and components $C_0, \dots, C_{n-1}$, whose indices $0, \dots, n-1$ are remainders modulo n, so the index next to $n-1$ is zero.

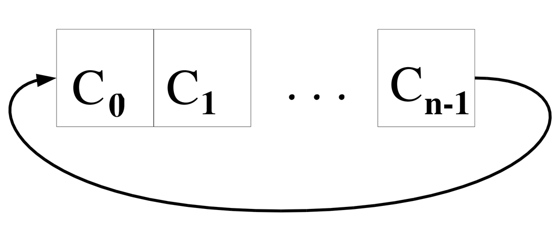

Figure 1: A circular C with $|C| = n$.

Transformations

As continuous time goes on, our sequence (finite or infinite) undergoes the following types of transformation:

$$\left.\begin{array}{l}\end{array}\right\} \quad (1)$$

- **Annihilation :** $(\oplus,\ominus) \to \Lambda$ and $(\ominus,\oplus) \to \Lambda$. If the states of the components with indices x and $x+1$ are different, both disappear with a rate α independently of the other components. The components $x-1$ and $x+2$ become neighbours. The length of the circular decreases by two.
- **Flip :** $\oplus \to \ominus$ and $\ominus \to \oplus$. This changes the state of one component with a rate β independently of the other components. The length of the circular does not change.
- **Mitosis :** $\oplus \to \oplus\oplus$ and $\ominus \to \ominus\ominus$. This duplicates one component with a rate γ independently of other components. The length of the cicular increases by one.

Main result

Our main results: Monte Carlo simulation and chaos approximation lead to similar pictures of ergodicity vs. non-ergodicity and growth vs. shrinking. (Figures 2 and 3.) Both of them suggest that with appropriate positive values of α, β and γ our processes have the following two properties:

- **Growth:** In the finite case the length of the circular tends to infinity with probability that tends to one, as the length of initial circular (consisting of several minuses) tends to ∞.
- **Non-ergodicity:** the infinite process is non-ergodic and the finite process keeps most of the time at two extremes, occasionally swinging from one to the other.

Our work was motivated by success and failure of [Toom], which considered infinite processes similar to ours with these differences: time was discrete (which we deem unimportant), flip was asymmetric, that is it turned minuses into pluses, but not vice versa (which also is unimportant for us since our initial configuration consists of minuses) and mitosis was absent (which is important). [Toom] proved some form of non-ergodicity for that process for α small enough: if the process started with "all minuses", the percentage of pluses always remained small. This was a success and it was improved in [RT.1] and studied numerically in [RT.2]. The failure of [Toom] was the impossibility to present a finite analog: in the absence of mitosis, length of the sequence decreased in average and the configuration degenerated. In our work this failure is removed.

Monte Carlo simulation

We approximate our infinite-space process with a Markov process with a countable set Ω of states, where Ω is the set of circulars of all lengths. The time t (that is, the number of iterations of our computer simulation) is discrete and at every time step at most one transformation of the list (1), chosen at random, takes place. Thus, in each individual experiment we obtain a randomly generated sequence of circulars and the circular obtained at time t is denoted by C^t. Its x-th component is denoted by C^t_x, where $x = 0, \dots, |C^t| - 1$.

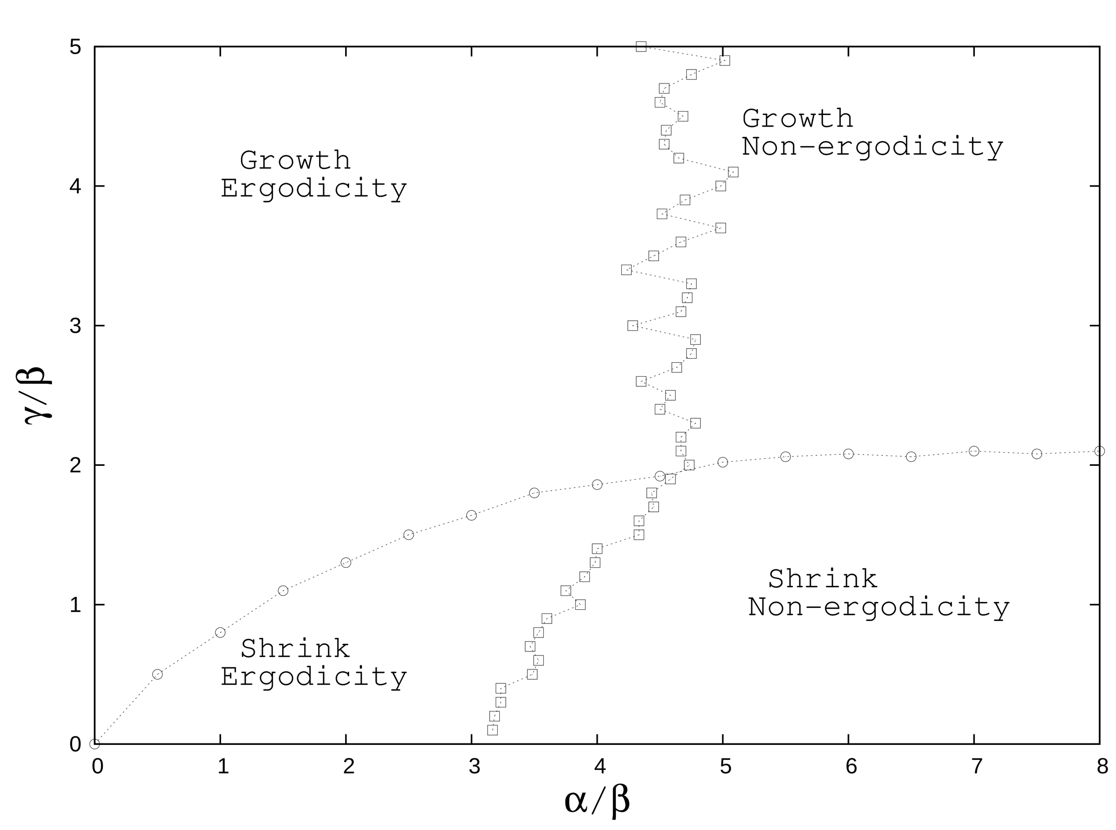

Figure 2: *White squares approximate the boundary between suggested ergodicity and suggested non-ergodicity. White balls approximate the boundary between suggested shrinking and suggested growth.*

Chaos approximation

Now let us describe the chaos approximation in our case. All the components of C^t are randomly permuted. Behavior of the resulting process essentially has only two parameters: quantity of pluses and quantity of minuses at time t, which we denote by $X(t)$ and $Y(t)$. When these quantities are large, we may approximatelly treat them as if they were real. In this approximation, we obtain a random process described by the following differential equations:

$$\left.\begin{aligned} \frac{dX(t)}{dt} &= -\beta \cdot X(t) + \beta \cdot Y(t) + \gamma \cdot X(t) - \alpha \cdot \frac{X(t)Y(t)}{X(t)+Y(t)}, \\ \frac{dY(t)}{dt} &= -\beta \cdot Y(t) + \beta \cdot X(t) + \gamma \cdot Y(t) - \alpha \cdot \frac{X(t)Y(t)}{X(t)+Y(t)}. \end{aligned}\right\}$$

Since the approximation is homogeneous, we may deal of other variables

$$S(t) = X(t) + Y(t) \quad \text{and} \quad B(t) = \frac{X(t) - Y(t)}{X(t) + Y(t)}$$

We call a number $B^* \in [-1,\ 1]$ a *fixed point* of this system if dB/dt equals zero at $B = B^*$. We say that a fixed point $B^* \in [-1,\ 1]$ attracts a point $B \in [-1,\ 1]$ if the process dB/dt starting at $B(0) = B$ tends to B^* when $t \to \infty$. Given a fixed point, we call its *basin of attraction* or just *basin* the set of points attracted by it.

We may write dS/dt as

$$\frac{d \ln S}{dt} = \gamma - \frac{\alpha}{2}(1 - B^2).$$

Let us denote by $G(B)$ the right side of $dlnS/dt$.

Given two positive functions f_1 and f_2 of $t \geq 0$, let us write $f_1 \asymp f_2$ if $f_1 = \mathcal{O}(f_2)$ and $f_2 = \mathcal{O}(f_1)$.

Lemma. Let $B(0) \in basin(B^*_i)$, where $i \in \{1, 2, 3\}$. Then:

- If $G(B^*_i) > 0$, then $\ln S(t) \asymp t$.
- If $G(B^*_i) = 0$, then $|\ln S(t)| = o(t)$.
- If $G(B^*_i) < 0$, then $-\ln S(t) \asymp t$.

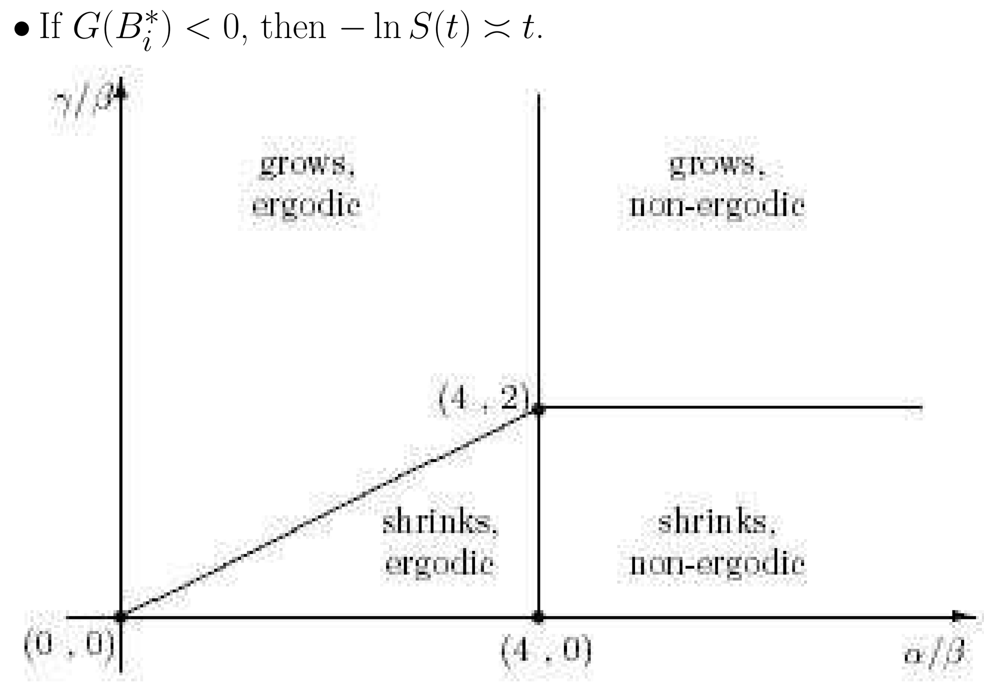

Figure 3: *Classification for $X(0) \neq Y(0)$. Compare this figure with figure 2.*

In the special case when $X(0) = Y(0)$ we have $B(t) = 0$ for all t. But zero is a fixed point, so the process is ergodic.

Conclusion

Our main purpose was to study a class of random processes, whose states were infinite in both directions sequences of pluses and minuses. At the same time we had to deal with analogous processes, whose states were finite sequences of pluses and minuses, which we called circulars. We studied these processes using two methods: Monte Carlo and chaos approximations. These methods led us to similar results and suggested that our processes can grow and be non-ergodic at the same time. So we may have found another example of 1-D non-ergodicity.

Acknowledgements

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