Toom's Stability Theorem in Continuous Time

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ABSTRACT This paper provides a continuous-time analogue of Toom's famous discrete-time stability criterion. Because of certain intrinsic differences between discrete and continuous time, simple analogues of Toom's result are not true, the main problem being that discrete-time models can undergo a spatial shift at each time step and continuous-time models cannot. In our main result, we show that once such shifts have been neutralized, the stability properties of a discrete-time model and its continuous-time analogue are the same. This result applies to a large class of models with finite range interactions in finite dimensions, including many for which the stability question was previously unanswered. Its proof uses an improved version of Toom's theorem that is found in [BG91]. We also obtain, as a byproduct of our analysis, an alternative criterion for stability in discrete time that is easy to check.

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18.1 A Brief Description of Toom's Theorem

Throughout this paper, the setting will be the state space

$$\Xi = \text{ the collection of all subsets of the } d\text{-dimensional integer lattice } \mathbb{Z}^d.$$

The elements of the lattice $\mathbb{Z}^d$ are called *sites*. A common interpretation of a state $A \in \Xi$ is that the sites in A are *occupied*, while the sites in the complement of A are *vacant*.

The discrete-time models covered by Toom's stability result are defined in terms of an operator that maps the state space Ξ to itself. Let N be a finite subset of $\mathbb{Z}^d$. A mapping $T\colon \Xi \to \Xi$ is a *Toom operator* with *neighborhood N* if it satisfies each of the following for all $x \in \mathbb{Z}^d$ and $A, B \in \Xi$:

(i) Monotonicity: if $A \subseteq B$, then $T(A) \subseteq T(B)$;
(ii) Translation-invariance: $T(A + x) = T(A) + x$;
(iii) Neighborhood N: $x \in T(A)$ if and only if $x \in T(A \cap [N + x])$;
(iv) Two-traps: $T(\emptyset) = \emptyset$ and $T(\mathbb{Z}^d) = \mathbb{Z}^d$.

Let

$$B(r) = \{x \in \mathbb{R}^d : |x| \leq r\}$$

for $r > 0$, where $|\cdot|$ is the ordinary Euclidean norm. A Toom operator T with neighborhood N has *range* r if $N \subseteq B(r)$. Property (iii) is also called the *finite range condition* when we don't want to be specific about the neighborhood N.

Because of the two-traps condition, the states $\emptyset$ and $\mathbb{Z}^d$ are fixed points of any Toom operator. In [Too80], Toom investigated the stability of these fixed points under 'one-sided' random perturbations. Since the two fixed points can be treated in a completely analogous manner, we concentrate our attention on the stability of the state $\mathbb{Z}^d$.

For each Toom operator T, there is a corresponding family of discrete-time Markov processes on Ξ, parameterized by a real number $\varepsilon \in [0, 1]$ known as the *error probability*. We typically let η_t, $t = 0, 1, 2, \ldots$, denote such a process. The dynamics are simple to describe. Let η_0 be an initial state. For times $t \geq 0$ and finite sets $A \subseteq \mathbb{Z}^d$,

$$P(A \subseteq \eta_{t+1} \mid \eta_t) = \begin{cases} (1-\varepsilon)^{\sharp A} & \text{if } A \subseteq T(\eta_t); \\ 0 & \text{otherwise.} \end{cases}$$

(Here and elsewhere, the symbol $\sharp$ means "cardinality of".)

An equivalent description of (η_t) can be made in terms of random *error sets*. Let $E_1, E_2, E_3, \ldots$ be a sequence of random subsets of $\mathbb{Z}^d$, whose distribution is determined by the condition that the events $\{x \in E_t\}$, $x \in \mathbb{Z}^d$, $t = 1, 2, \ldots$, are independent and each have probability ε. Given an initial state η_0, define η_t inductively by

$$\eta_{t+1} = T(\eta_t) \setminus E_{t+1}, n = 1, 2, 3, \ldots. \tag{18.1.1}$$

We may informally describe the process (η_t) as follows. If, at some time t, the set of occupied sites is A, then, in the absence of errors, the set of occupied sites at time $t+1$ would be $T(A)$. But errors occur independently with probability ε at the sites in $T(A)$. As a result, the sites in $T(A) \cap E_{t+1}$ are vacant at time $t+1$, rather than occupied. The errors are *one-sided*, in the sense that the sites in $\mathbb{Z}^d \setminus T(A)$ are unaffected: a site in $\mathbb{Z}^d \setminus T(A)$ is vacant at time $t+1$, whether or not it happens to be part of the error set E_{t+1}.

The types of errors described in the preceding paragraph are those that were considered by Toom, and they suffice for the purposes of this section. In the proof of the main result in Section 18.4, we will use more general kinds of one-sided error sets E_t. Toom's results were extended to models with such error sets in [BG91].

We are interested in what happens when the initial state is $\eta_0 = \mathbb{Z}^d$ and ε is small but positive. There is a standard argument (found in [Lig85], for example) that shows that because of the monotonicity condition, the quantity $P[0 \in \eta_n]$ decreases in n, so the following limit exists:

$$\pi(\varepsilon) = \lim_{n \to \infty} P[0 \in \eta_n].$$

Furthermore, the monotonicity condition also implies that $\pi(\varepsilon)$ is monotonically decreasing as a function of ε. We say that T is *stable* at $\mathbb{Z}^d$ if $\lim_{\varepsilon \searrow 0} \pi(\varepsilon) = 1$. Note that the two-traps condition implies that $\pi(0) = 1$.

Toom proved that the following condition is necessary and sufficient for stability at $\mathbb{Z}^d$:

Toom's Eroder Condition

For every finite set $D \subseteq \mathbb{Z}^d$, there exists an n such that $T^n(D^c) = \mathbb{Z}^d$.

Any Toom operator T satisfying this condition is called an *eroder*. Toom's stability theorem says that T is stable at $\mathbb{Z}^d$ if and only if T is an eroder.

The eroder condition is quite beautiful, because of its simplicity. Nevertheless, in general it can be difficult to check, since it involves *every* finite set $D \subseteq \mathbb{Z}^d$. We will see (Theorem 18.2.1 and its corollary) that there are other more explicit criteria that involve analyzing Toom operators in terms of certain spatial shifts. By way of preparation, we conclude this section with a few definitions and general comments about the way in which spatial shifts affect Toom operators.

Given a Toom operator T and a point $z \in \mathbb{Z}^d$, the mapping $T + z$ is also a Toom operator, where $(T + z)(A) = T(A) + z$. The Toom operator $T + z$ is a *spatial shift* of T. It is obvious that the stability of a Toom operator is not affected by spatial shifts. It is also clear from Toom's eroder condition that a Toom operator T is stable if and only if the n-fold iterate T^n is stable for some (every) positive integer n. These facts motivate the following definition: we say that two Toom operators T_1 and T_2 are *shift-equivalent* if there exists a site $z \in \mathbb{Z}^d$ and positive integers m and n such that

$$(T_1)^m = (T_2)^n + z.$$

Note that shift-equivalence is an equivalence relation. In the next section, we will show that within each equivalence class, there is at least one Toom operator with particularly nice properties.

The main purpose of this paper is to prove a continuous-time analogue of Toom's stability theorem. As we will see in Section 18.3, for each Toom operator T there is a natural choice for a parameterized family of continuous-time Markov processes. However, there is an important difference between continuous time and discrete time. In discrete time, the model with $\varepsilon = 0$ is deterministic, while the analogous model in continuous time is not. One consequence is that if two different Toom operators are shift-equivalent, then the two models corresponding to them are trivially related to each other in discrete time, but not in continuous time. Indeed, we will see that there are examples in continuous time in which the two models do not even have the same stability properties.

18.2 The Eroder Condition and Spatial Shifts

One purpose of this section is to provide the context for our main result, Theorem 18.3.1; the concepts developed here appear in the hypotheses of Theorem 18.3.1, and the results given here make it clear that Theorem 18.3.1 is an appropriate analogue of Toom's stability theorem.

Another purpose of this section is to focus attention on certain aspects of the eroder condition that are often overlooked. In particular, Theorem 18.2.1 gives conditions that are equivalent to the eroder condition. These conditions are not as elegant as the eroder condition, but they have the advantage of being easier to verify and easier to apply. The essential idea behind one of these conditions, Equation (18.2.2) below, is due to Toom.

Throughout this discussion, we assume that we have been given a Toom operator T in d dimensions with finite range r.

In order to understand the role that spatial shifts play, it is natural to look at how T operates on half-spaces. For vectors $v, w \in \mathbb{R}^d$, let

$$\mathcal{H}(v, w) = \{x \in \mathbb{R}^d : \langle x - w, v\rangle < 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product. Thus $\mathcal{H}(v, w)$ is the open half-space in $\mathbb{R}^d$ that is uniquely determined by three conditions: (i) its boundary is oriented perpendicularly to v, (ii) its boundary contains the point w, (iii) v points towards the closed half-space $\mathcal{H}(v, w)^c$. It will also be useful to have some notation for the boundary of $\mathcal{H}(v, w)$,

$$\mathcal{P}(v, w) = \text{ the boundary of } \mathcal{H}(v, w),$$

and for the *lattice half-spaces*,

$$H(v, w) = \mathcal{H}(v, w) \cap \mathbb{Z}^d.$$

Let $\mathbf{0}$ denote the origin in $\mathbb{R}^d$, and let

$$R = \mathbb{R}^d \setminus \mathbf{0} \quad \text{and} \quad S = \mathbb{Z}^d \setminus \mathbf{0}.$$

In the following result, we introduce a function with domain R. This function is surprisingly useful, as it gives us a convenient formula (Equation (18.2.1)) for the way in which T operates on lattice half-spaces.

Proposition 18.2.1 *Let $\alpha: R \to \mathbb{R}$ be defined by the formula*

$$\alpha(v) = \sup\{a \in \mathbb{R} : \mathbf{0} \in T(H(v, -av))\}.$$

Then for $v \in R$ and $w \in \mathbb{R}^d$,

$$T(H(v, w)) = H(v, w + \alpha(v)v). \tag{18.2.1}$$

Furthermore, the function α has the following three properties: (i) $\alpha(cv) = \alpha(v)/c$ *for all $c > 0$ and all $v \in R$;* (ii) *α is uniformly continuous on any closed subset of R;* (iii) *α takes only rational values when restricted to S.*

PROOF. We first prove (18.2.1). By the monotonicity of T and the fact that $\mathcal{H}(v, w)$ is open,

$$\mathbf{0} \in T(H(v, (-\alpha(v) + s)v)) \text{ iff } s > 0.$$

Since T has neighborhood N, it follows that the boundary of $\mathcal{H}(v, -\alpha(v)v)$ contains a (not necessarily unique) site $w(v) \in N$. So $\mathcal{H}(v, -\alpha(v)v) = \mathcal{H}(v, w(v))$. Using translation-invariance and monotonicity, it is now easy to first prove (18.2.1) for $w = -w(v)$, and then for all vectors $w \in \mathbb{Z}^d$. To extend to all $w \in \mathbb{R}^d$, use monotonicity, finiteness of range, and the simple fact that for every such w, there exists a sequence $w_1, w_2, \ldots \in \mathbb{Z}^d$ such that

$$\mathcal{H}(v, w) = \bigcup_{k=1}^{\infty} \mathcal{H}(v, w_k).$$

Property (i) is immediate from the definition of α. Property (ii) is an easy consequence of the finite range property of T. Finally, it is easy to see from the definitions that if $w(v)$ is as in the preceding paragraph, then $\alpha(v) = -\langle w(v), v\rangle$. Thus Property (iii) also holds. □

The function α is called the *speed coefficient* of T. We will eventually see how the speed coefficient can be used to determine whether T is an eroder.

In what follows, we will often be interested in how T acts on unions of half-spaces, so it will be convenient to have some special notation for such sets. Given a set V of vectors in R and a function $\varphi\colon V \to \mathbb{R}^d$, we let

$$\mathcal{H}(V, \varphi) = \bigcup_{v \in V} \mathcal{H}(V, \varphi(v)) \quad \text{and} \quad H(V, \varphi) = \mathcal{H}(V, \varphi) \cap \mathbb{Z}^d.$$

It will be common for our choice of the function φ to depend in some way on the identity function on R. We will use the notation

$$I = \text{ the identity function on } R.$$

For example, a common choice for φ will be the function αI. Note that (18.2.1) and monotonicity imply the following:

Proposition 18.2.2 *If $V \subseteq R$, then*

$$T(H(V, \varphi)) \supseteq H(V, \varphi + \alpha I),$$

for any function $\varphi\colon V \to \mathbb{R}^d$.

A typical application of the preceding result is as follows: let D be a finite subset of $\mathbb{Z}^d$, choose $c > 0$ so that $H(V, -cI) \subseteq D^c$. If there exists a positive integer n such that $H(V, n\alpha I - cI) = \mathbb{Z}^d$, then by monotonicity and Proposition 18.2.2, $T^n(D^c) = \mathbb{Z}^d$. So the proposition makes it possible to connect the eroder condition to the speed coefficient.

Of particular interest to us is the set $\mathcal{H}(R, \alpha I)$, which is the set obtained by shifting each half-space $\mathcal{H}(v, \mathbf{0})$ by the amount $\alpha(v)|v|$ in the v direction, and then taking the union over all vectors $v \in R$. We will see that T is an eroder if and only if $\mathcal{H}(R, \alpha I) = \mathbb{R}^d$. It will be convenient to be able to express this condition in terms of a nice subset of R. The next somewhat technical proposition and its corollary help us do just that. In preparation, we introduce the notation

$$L(v) = \text{ the linear span of } \mathcal{P}(v, \mathbf{0}) \cap B(2r) \cap \mathbb{Z}^d,$$

where we assume that T has range r, and that $r \geq 1$.

Lemma 18.2.1 *Let*

$$U = \{u \in R\colon L(u) \text{ has dimension } d-1\}.$$

Then $\mathcal{H}(R, \alpha I) = \mathcal{H}(U, \alpha I)$.

PROOF. See Section 18.5. □

If $L(u)$ has dimension $d-1$, then u is the solution of a set of $d-1$ linear equations of the form $\langle u, v_i\rangle = 0$, where $v_1, \ldots, v_{d-1}$ are linearly independent vectors with integer coordinates. Thus, the method of Gaussian elimination makes it clear that some nonzero scalar multiple of u lies in S. By Property (i) in Proposition 18.2.1, $\mathcal{H}(cu, \alpha(cu)cu) = \mathcal{H}(u, \alpha(u)u)$ for all $c > 0$. Therefore, the following corollary is an immediate consequence of Lemma 18.2.1. It states that U may be replaced by a certain finite set W. Note that, like U, this set depends only on the dimension d and the range r.

Corollary 18.2.1 *Let*

$$W = \{w \in U \cap S\colon \text{the coordinates of } w \text{ have no common integer divisor}\}.$$

Then $\mathcal{H}(W, \alpha I) = \mathcal{H}(R, \alpha I)$.

We have one more technical result. It is a geometric result about unions and intersections of open half-spaces that does not really depend on the properties of Toom operators or their speed coefficients. However, to make the result easier to use in our context, the speed coefficient appears in its statement.

Lemma 18.2.2 *Suppose* $\mathcal{H}(W, \alpha I) = \mathbb{R}^d$, *and let* V *be a minimal subset of* W *such that* $\mathcal{H}(V, \alpha I) = \mathbb{R}^d$. *Then* $\mathcal{H}(V, -\alpha I)^c$ *contains at least one point with rational coordinates.*

PROOF. See Section 18.5. □

We are now ready to define two special classes of Toom operators. Let T be a Toom operator with speed coefficient α. If $\alpha(v) \leq 0$ for all vectors $v \in W$, then T is an *expander*. If there exists a set $V \subseteq W$ such that α is nonnegative on V and $\mathcal{H}(V, \alpha I) = \mathbb{R}^d$, then T is a *shrinker*. Like the term "eroder", the words "expander" and "shrinker" are intended to suggest the way in which T affects sets of vacant sites. Theorem 18.2.1 states that every Toom operator is shift-equivalent to either an expander or a shrinker, and that a Toom operator is an eroder if and only if it is shift-equivalent to a shrinker.

Theorem 18.2.1 *A Toom operator* T *with speed coefficient* α *is shift-equivalent to a shrinker if*

$$\mathcal{H}(W, \alpha I) = \mathbb{R}^d. \tag{18.2.2}$$

Otherwise, it is shift-equivalent to an expander. Furthermore, T *is an eroder if and only if* (18.2.2) *holds.*

PROOF. First assume that $\mathcal{H}(W, \alpha I) = \mathbb{R}^d$. By Lemma 18.2.2, there exists a finite set $V \subseteq W$ such that $\mathcal{H}(V, \alpha I) = \mathbb{R}^d$ and such that $\mathcal{H}(V, -\alpha I)^c$ contains a point x with rational coordinates. Let n be a positive integer such that nx has integer coordinates, and consider the Toom operator $T' = T^n - nx$. It is easy to see that $\mathbf{0} \in \mathcal{H}(V, -\alpha' I)^c$, where α' is the speed coefficient of T'. It follows that $\alpha' \geq 0$ on V. Since $\mathcal{H}(V, \alpha I) = \mathbb{R}^d$, it is obvious that $\mathcal{H}(V, \alpha' I) = \mathbb{R}^d$. So T' is a shrinker, as desired.

Next we show that if the set $A = \mathcal{H}(W, \alpha I)^c$ is nonempty, then T is shift-equivalent to an expander. By construction, A is closed and convex. Since T has finite range, A is bounded. Therefore, each extreme point of A equals the intersection of hyperplanes $\mathcal{P}(v, \alpha(v)v)$, taken over some set of vectors $v \in W$. As in the proof of Lemma 18.2.2, it follows that A must contain at least one point x with rational coordinates. Let n be a positive integer such that nx has integer coordinates, and consider the operator $T' = T^n - nx$. As in the previous step, it is easy to see that $\mathcal{H}(W, \alpha' I)^c$ contains the origin, so $\alpha' \leq 0$ on W, and T' is an expander, as desired.

Since the eroder property is preserved under shift-equivalence, it is enough to show that no expander is an eroder, and that every shrinker is an eroder. We first show that every shrinker is an eroder.

Let T be a shrinker, with speed coefficient α. By definition there exists a finite set V such that $\alpha \geq 0$ on V and $\mathcal{H}(V, \alpha I) = \mathbb{R}^d$, from which it follows that there exists a vector $v \in V$ such that $\alpha(v) > 0$. It is easy to see from these facts that for every $c > 0$, there exists a positive integer n such that $\mathcal{H}(V, (n\alpha - c)I) = \mathbb{R}^d$. By Propositions 18.2.1 and 18.2.2, $T^n(H(V, -cI)) = \mathbb{Z}^d$. It follows immediately from monotonicity that T is an eroder.

It remains to show that if T is an expander, then T is not an eroder. By Corollary 18.2.1 and the definition of expander, $\alpha(v) \leq 0$ for all $v \in R$. We now refer back to [Too80], where it is shown (using different terminology of course) that the speed coefficient of an eroder must take at least one positive value. Thus, T cannot be an eroder. □

As stated earlier, (18.2.2) is a condition that can be readily checked. We briefly describe here an explicit algorithm for doing so. First, determine the set W. One way to do this is to use brute force: look at all possible subsets of $B(2r) \cap S$ of size $d - 1$ to see which ones determine a $d - 1$-dimensional hyperplane through the origin, and for those that do, use algebra to find a vector with integer coordinates that is perpendicular to the hyperplane. Once W is determined, calculate $\alpha(w)$ for each $w \in W$ by comparing the two lattice half-spaces $H(w, \mathbf{0})$ and $T(H(w, \mathbf{0}))$. Then determine (using methods of linear programming) the solution set of the system of equations

$$\langle x - \alpha(w)w, w \rangle \geq 0, w \in W. \tag{18.2.3}$$

This solution set equals $\mathcal{H}(W, \alpha I)^c$, so (18.2.2) holds if and only if (18.2.3) has no solution. Thus, we have

Corollary 18.2.2 *A Toom operator T is stable if and only if (18.2.3) has no solution.*

When a solution to (18.2.3) exists, any member of the solution set with rational coefficients can be used, as in the proof of Theorem 18.2.1, to find an expander that is shift-equivalent to T. Otherwise, Lemma 18.2.2 and the proof of Theorem 18.2.1 indicate an explicit method for finding a shrinker that is shift-equivalent to T.

18.3 A Class of Continuous-Time Systems

In continuous time, we are concerned with certain Markov processes $(\xi_t, t \geq 0)$, with state space Ξ. They are defined in terms of *birth rates* and *death rates.* For each $x \in \mathbb{Z}^d$, let β_x denote the birth rate at the site x and δ_x denote the death rate at x. Each such birth or death rate is a function from the state space Ξ to the nonnegative real numbers. The following formulas give the meanings of these rates:

$$P(x \in \xi_{t+h} \mid \xi_t \text{ and } x \notin \xi_t) = \beta_x(\xi_t)h + o(h),$$

$$P(x \notin \xi_{t+h} \mid \xi_t \text{ and } x \in \xi_t) = \delta_x(\xi_t)h + o(h),$$

and

$$P(\text{a change occurs at both } x \text{ and } y \text{ during } (t, t+h] \mid \xi_t) = o(h).$$

Thus, the rates are instantaneous expected rates of change, given the current state of the system, with the birth rates being relevant at vacant sites and the death rates being relevant at occupied sites. Processes meeting the above description are sometimes called *Markovian spin-flip systems.* For the technical details of the construction of such processes, see a general reference such as [Lig85] or [Dur88].

Under very broad conditions (which are satisfied for all of the models considered here), the state $\mathbb{Z}^d$ is absorbing if and only if $\delta_x(\mathbb{Z}^d) = 0$ for all $x \in \mathbb{Z}^d$. In this case, the appropriate notion of stability at $\mathbb{Z}^d$ involves perturbing the death rates by adding a nonnegative constant ε. The parameter ε is called the *error rate.* In what follows, we will typically use the notation $(\xi_t, t \geq 0)$ for the parameterized Markovian particle system with birth rates β_x, death rates $\delta_x + \varepsilon$, and initial state $\mathbb{Z}^d$.

The formal definition of stability in continuous time is similar to that in discrete time: the state $\mathbb{Z}^d$ is *stable* if

$$\liminf_{\varepsilon \searrow 0} \pi(\varepsilon) = 1,$$

where

$$\pi(\varepsilon) = \liminf_{t \to \infty} P[0 \in \xi_t].$$

We use the lim inf in these expressions to accommodate models with non-attractive rates. (Rates are *attractive* if they satisfy the following condition for all $x \in \mathbb{Z}^d$

and $A, B \subseteq \Xi$: if $A \subseteq B$, then $\beta_x(A) \leq \beta_x(B)$ and $\delta_x(A) \geq \delta_x(B)$.) If the rates are attractive, standard arguments ensure that the lim inf can be replaced by a limit in both of the expressions above involving $\pi(\varepsilon)$.

For the remainder of this paper, we will consistently make the following three assumptions about sets of birth and death rates (β_x, δ_x).

For all $A, B \in \Xi$ and $x, y \in \mathbb{Z}^d$:

(i) Weak attractiveness: if $A \subseteq B$, then $\beta_x(A) > 0 \Rightarrow \beta_x(B) > 0$ and $\delta_x(B) > 0 \Rightarrow \delta_x(A) > 0$;
(ii) Translation-invariance: $\beta_x(A) = \beta_{x+y}(A+y)$ and $\delta_x(A) = \delta_{x+y}(A+y)$;
(iii) Finite range property: there exists a finite set $N \subseteq \mathbb{Z}^d$ that does not depend on x or A such that $\beta_x(A) = \beta_x(A \cap (N+x))$ and $\delta_x(A) = \delta_x(A \cap (N+x))$.

The second and third assumptions guarantee sufficient regularity so that there are no difficulties with the construction of the corresponding Markovian interacting particle systems.

We are now ready to show how to relate continuous-time systems with discrete-time systems. Given a set of rates (β_x, δ_x) satisfying the above three conditions, define an operator $T: \Xi \to \Xi$ by the following formula:

$$T(A) = \{x \in A: \delta_x(A) = 0\} \cup \{x \in A^c: \beta_x(A) > 0\}. \tag{18.3.1}$$

The rates (β_x, δ_x) are called *Toom rates* if T is a Toom operator. If (β_x, δ_x) are Toom rates, then the operator defined in (18.3.1) is called the Toom operator *corresponding to* (β_x, δ_x). Toom rates are called *canonical* if they take only the values 0 and 1. Examples of models with Toom rates are given at the end of this section. These include d-dimensional versions of the ordinary contact process, the sexual contact process, and the majority vote model.

If we ignore the irrelevant values of $\delta_x(A)$ for $x \notin A$ and of $\beta_x(A)$ for $x \in A$, it is easy to see that (18.3.1) determines a one-to-one correspondence between Toom operators and canonical Toom rates. The "inverse" of (18.3.1) is

$$\beta_x(A) = \begin{cases} 1 & \text{if } x \in T(A \setminus \{x\}) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \notin T(A \cup \{x\}) \\ 0 & \text{otherwise,} \end{cases}$$

for $x \in \mathbb{Z}^d$ and $A \subseteq \mathbb{Z}^d$.

It is also easy to see from the four properties in the definition of Toom operator that (β_x, δ_x) are Toom rates if and only if they are weakly attractive, translation-invariant, finite range, and also satisfy the following two properties for $x \in \mathbb{Z}^d$ and $A, B \in \Xi$:

(iv) Two-traps: $\beta_x(\emptyset) = \delta_x(\mathbb{Z}^d) = 0$;
(v) Exclusiveness: $\beta_x(A \setminus \{x\})\delta_x(A \cup \{x\}) = 0$ for all $x \in \mathbb{Z}^d$ and $A \in \Xi$.

Just as in discrete time, Property (iv) implies that $\emptyset$ and $\mathbb{Z}^d$ are absorbing states. To see the reason for the Property (v), note that because of the monotonicity of T,

$$x \in T(A \setminus \{x\}) \quad \Rightarrow \quad x \in T(A \cup \{x\}).$$

This implication and its contrapositive immediately give Property (v). Note that canonical Toom rates are attractive, rather than just weakly attractive.

Here is the statement of our main result. The proof is given in Section 18.4.

Theorem 18.3.1 *Let (β_x, δ_x) be Toom rates, and let T be the corresponding Toom operator, as defined by (18.3.1). A continuous-time system with these rates is stable at $\mathbb{Z}^d$ if T is a shrinker. It is not stable at $\mathbb{Z}^d$ if T is an expander.*

A standard comparison argument using "coupling" (see [Lig85]) gives the following corollary:

Corollary 18.3.1 *Let (β_x, δ_x) be rates that satisfy the assumptions* (i), (ii), *and* (iii) *given above. Assume that $\delta_x(\mathbb{Z}^d) = 0$ for all $x \in \mathbb{Z}^d$. Let T be defined in terms of these rates as in (18.3.1). The continuous-time system with these rates is stable at $\mathbb{Z}^d$ if there exists a shrinker T' such that $T'(A) \subseteq T(A)$ for all $A \in \Xi$. It is not stable at $\mathbb{Z}^d$ if there exists an expander T' such that $T(A) \subseteq T'(A)$ for all $A \in \Xi$.*

We now give several examples. In each case, a class of models (or a specific model) is described in terms of the dimension d, a neighborhood set N, a formula for the birth rate $\beta_{\mathbf{0}}$, and a formula for the unperturbed death rate $\delta_{\mathbf{0}}$. The remaining rates β_x and δ_x can then be easily determined by applying translation-invariance.

Example 18.3.1 (Pure-birth models) A *pure-birth* model is one with weakly attractive, translation-invariant, finite-range rates (β_x, δ_x) such that $\delta_x \equiv 0$ for all x. Theorem 18.3.1 gives such a nice result in this case, that it is worth stating as a corollary. As a general stability criterion for pure-birth models, this result is new. After its statement and easy proof, we give several specific applications.

Corollary 18.3.2 *A pure-birth model with birth rates (β_x) and neighborhood N is stable at $\mathbb{Z}^d$ if and only if*

$$\beta_{\mathbf{0}}(A) > 0 \qquad (18.3.2)$$

for some set $A \subseteq N$ whose convex hull in $\mathbb{R}^d$ does not contain the origin.

PROOF. The most important case is when the rates are Toom rates. Since the death rate is identically 0, the corresponding Toom operator has a nonnegative speed coefficient. So it is an expander if this speed coefficient is identically 0, and otherwise it is a shrinker. It is now easy to see that (18.3.2) is a necessary and sufficient condition for T to be a shrinker, and the desired result follows from Theorem 18.3.1.

Since the death rates are identically equal to 0, it is easy to see that the rates can only fail to be Toom rates if the birth rates are strictly positive, in which case (18.3.2) obviously holds. It is not hard to prove stability directly in this case, but stability is also an immediate consequence of Corollary 18.3.1 (just let the comparison operator T' be any shrinker). □

The most famous class of pure-birth models consists of the *contact processes*, for which $\beta_{\mathbf{0}}(A) > 1$ when $A \cap N$ is nonempty, and $\beta_{\mathbf{0}}(A) = 0$ otherwise. The two most common contact processes are defined by the birth rate formulas $\beta_{\mathbf{0}}(A) = \sharp(A \cap N)$ and $\beta_{\mathbf{0}}(A) = \min\{1, \sharp(A \cap N)\}$. The second formula gives canonical Toom rates. For any contact process with neighborhood N, the stability condition (18.3.2) is equivalent to the condition that $N \setminus \{\mathbf{0}\} \neq \emptyset$. This is a known result. For an introduction to the extensive body of research concerning contact processes, see [Lig85] and [Dur88].

A second important class of pure-birth models consists of the *sexual contact processes*. These are similar to the contact processes, except that $A \cap N$ must contain at least 2 sites in order for $\beta_{\mathbf{0}}(A)$ to be positive. In the literature, sexual contact processes are usually assumed to have canonical Toom rates, so that

$$\beta_{\mathbf{0}}(A) = \begin{cases} 1 & \text{if } \sharp(A \cap N) \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

The most studied example is the sexual contact process in 2 dimensions with the *NEC neighborhood* $N = \{(0,0), (1,0), (0,1)\}$.

For the sexual contact process with arbitrary neighborhood N, the stability condition (18.3.2) is equivalent to the condition that $N \setminus \{\mathbf{0}\}$ contain at least two sites that do not lie on a straight line through the origin. This result is also known, but the proof, which is due to Durrett and Gray, is unpublished. Sexual contact processes have many interesting stability properties, which are discussed in [DG86], [Che92], and [Che94].

In order to find a completely new application for Corollary 18.3.2, we need to look at processes where a birth can only occur at x if more than three sites in $N + x$ are occupied. The simplest case is the following variation on the sexual contact process: let $d = 3$, $N = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$, $\delta_x \equiv 0$, and

$$\beta_x(A) = \begin{cases} 1 & \text{if } N + x \subseteq A \cup \{x\} \\ 0 & \text{otherwise.} \end{cases}$$

This birth rate satisfies (18.3.2) with $A = N \setminus \{\mathbf{0}\}$. The stability at $\mathbb{Z}^3$ of the system with these rates was previously unknown. The reader can easily create other examples along these lines.

The remaining examples have nonzero death rates.

Example 18.3.2 (Majority vote models) For this class of examples, we assume that $\sharp N$ is odd. Let d be arbitrary. The birth rate $\beta_{\mathbf{0}}(A)$ is positive if $\sharp(A \cap N) > \sharp N/2$ and 0 otherwise. The unperturbed death rate $\delta_{\mathbf{0}}(A)$ is positive if $\sharp(A \cap N) < \sharp N/2$

and 0 otherwise. Usually, the rates are assumed to be canonical Toom rates, so that they equal 1 whenever they are not equal to 0. But there are good reasons to allow values other than 0 and 1 in these models (see the "double stability" example below).

If the neighborhood is symmetric, such as for the *nearest neighbor majority vote process*, for which $N = \{y: |y| \leq 1\}$, the corresponding Toom operator is an expander. Thus, majority vote models with symmetric neighborhoods are unstable at $\mathbb{Z}^d$. For many asymmetric neighborhoods, such as the NEC neighborhood in $\mathbb{Z}^2$ defined in the previous example, the corresponding Toom operator is a shrinker, so stability holds. Note that a majority vote process with the NEC neighborhood has the same birth rates as a sexual contact process with the NEC neighborhood (but not the same death rates).

The instability of symmetric neighborhood majority vote processes and the stability of NEC neighborhood majority vote processes are known. For many other choices of the neighborhood, Theorem 18.3.1 settles the stability question for the first time.

An instructive example for which Theorem 18.3.1 gives no information is the one-dimensional majority vote model with $N_x = \{x, x+1, x+2\}$. The corresponding Toom operator is neither a shrinker nor an expander (it is shift-equivalent to an expander). The stability properties of this model are unknown.

Example 18.3.3 (The double stability of the sexual contact process.) We have already explained that our main result implies directly that the sexual contact process with the NEC neighborhood in 2 dimensions is stable at $\mathbb{Z}^2$. As discussed in [DG86], this model is also stable at $\emptyset$ for any positive ε. This behavior contrasts with that of the ordinary contact process, which is not stable at $\emptyset$ for small positive $\varepsilon > 0$. (In an interacting particle system with $\emptyset$ as an absorbing state, stability at $\emptyset$ means that if the initial state is $\emptyset$ and a small positive quantity is added to the birth rate, then the probability that the origin is vacant at time t stays close to 1 as $t \to \infty$.) We explain here how the stability at $\emptyset$ of the sexual contact process follows from Corollary 18.3.1.

Let (ξ_t) be the sexual contact process with the NEC neighborhood in 2 dimensions, with error rate $\varepsilon > 0$. Consider the process (ζ_t) defined by $\zeta_t = (\xi_t)^c$. The death rates of this model are the death rates of an NEC majority vote model, and its birth rates equal the constant ε. This model does not have Toom rates. However, if T is defined as in (18.3.1), then it is easy to see that T satisfies the hypothesis of the first conclusion in Corollary 18.3.1, with T' being the Toom operator corresponding to an NEC majority vote model. So (ζ_t) is stable at $\mathbb{Z}^d$. Now transform back to see that (ξ_t) is stable at $\emptyset$. The original proof of this result, which is due to Durrett and Gray, is unpublished. See [DG86] for further details.

The next example illustrates a limitation of our results. Toom's stability theorem suffers from a similar limitation.

Example 18.3.4 (Stability without Toom rates) In [GG82], a necessary and sufficient stability condition is obtained for the class of models with $d = 1$ and

$N = \{-1, 0, 1\}$ whose rates are attractive, translation-invariant and satisfy the two-traps condition. But it is not assumed that the rates satisfy the exclusiveness condition, so that, for instance, both $\beta_0(\{1\})$ and $\delta_0(\{0, 1\})$ might be nonzero. When the rates are not exclusive, there is no corresponding Toom operator.

The necessary and sufficient condition for stability in [GG82] is that

$$\beta_0(\{1\}) + \beta_0(\{-1\}) > \delta_0(\{0, 1\}) + \delta_0(\{-1, 0\}).$$

Even when the rates are exclusive, the results in the present paper will not apply unless one of the two sides of this inequality equals 0, since otherwise, the corresponding Toom operator is neither a shrinker nor an expander.

We conclude this section with two examples showing that models with Toom rates need not have the same stability properties as their corresponding Toom operators, if these Toom operators are not shift-equivalent to shrinkers or expanders. These examples illustrate why shift-equivalence does not play the same role in continuous time as it does in discrete time. Because of space limitations, the reader will need to verify the assertions made in these two examples. (This would make a nice warm-up project for a talented graduate student.)

Example 18.3.5 (Unstable Toom rates corresponding to an eroder) Let $d = 2$. For each integer m, let $N^m = \{(m, k)\colon k = -n, -n+1, \ldots, n-1, n\}$, where n is a sufficiently large positive integer. For $x \in \mathbb{Z}^d$ and $A \subseteq \mathbb{Z}^d$, define

$$\beta_x(A) = \begin{cases} 1 & \text{if } (N^{-2} \cup N^{-3} + x) \subseteq A \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta_x(A) = \begin{cases} 1 & \text{if } \{x - (1, 0), x - (2, 0)\} \subseteq A^c \\ 0 & \text{otherwise.} \end{cases}$$

We leave it to the reader to check that we have defined a set of canonical Toom rates corresponding to a Toom operator T, whose speed coefficient satisfies $\alpha((1, 0)) = 2$, $\alpha((-1, 0)) = -1$, and $\alpha((0, 1)) = \alpha((0, -1)) = 0$. It is easy to see from these values of $\alpha(\cdot)$ that $T + (-1, 0)$ is a shrinker, so T itself is an eroder.

In discrete time, the half-plane $H((1, 0), (0, 0))$ moves to the right at speed 2, which is enough to make T into an eroder. In continuous time, the corresponding movement of occupied sites is much slower when n is large, due to the fact that it takes a long time for a large vertical column of vacant sites to become occupied. On the other hand, horizontal blocks of vacant sites spread relatively easily to the right. With a little work, it can be shown that the vacant sites "win out" when n is large, destroying stability at $\mathbb{Z}^2$.

The next example uses the main idea of the preceding example, but in reverse. For technical reasons, the rates are somewhat more complicated than in the preceding example.

Example 18.3.6 (Stable Toom rates corresponding to a noneroder) Let $d = 2$. For each integer m, let $N_+^m = \{(m, k)\colon k = 0, 1, \ldots, n-1, n\}$, where n is a sufficiently large positive integer. For $x \in \mathbb{Z}^d$ and $A \subseteq \mathbb{Z}^d$, define

$$\delta_x(A) = \begin{cases} 1 & \text{if } (N_+^{-2} \cup N_+^{-3} + x - (0, j)) \subseteq A^c \text{ for some } j = 0, 1, \dots, n \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\beta_x(A) = \begin{cases} 1 & \text{if } \{x - (1, 0), x - (2, 0)\} \subseteq A \\ 0 & \text{otherwise.} \end{cases}$$

We leave it to the reader to check that we have defined a set of canonical Toom rates corresponding to a Toom operator T, whose speed coefficient satisfies $\alpha((1, 0)) = 1, \alpha((-1, 0)) = -2$, and $\alpha((0, 1)) = \alpha((0, -1)) = 0$. Further investigation of the values of $\alpha(\cdot)$ shows that $T + (-1, 0)$ is an expander, so T is not an eroder.

It can be shown that the continuous-time system with unperturbed rates (β_x, δ_x) is stable at $\mathbb{Z}^d$ for sufficiently large n. The choice of n is made so that a vertical "wall" of vacant sites spreads very slowly to the right. Blocks of occupied sites spread to the right at speed 1 in the unperturbed system. For a sufficiently small noise rate $\varepsilon > 0$, these two facts can be used to show that the perturbed system with rates $(\beta_x, \delta_x + \varepsilon)$ prevents vacant regions from growing very large, so that the probability that the origin is occupied at time t stays close to 1 as $t \to \infty$. One way to prove this fact is to use the comparison techniques found in our proof of Theorem 18.3.1.

18.4 Proof of Theorem 18.3.1

The proof of our main theorem will be carried out by making a comparison between a continuous-time process with Toom rates and a discrete-time model whose Toom operator is related to the Toom operator determined by the rates. To define this discrete-time model, we will need some terminology.

Given a Toom operator T, we define two auxiliary operators T_β and T_δ:

$$T_\beta(A) = A \cup T(A) \text{ and } T_\delta(A) = A \cap T(A), \quad A \subseteq \mathbb{Z}^d.$$

Proposition 18.4.1 *Let T_β, T_δ be defined as above in terms of a Toom operator T. Then T_β and T_δ are themselves Toom operators, and they satisfy*

$$T_\beta(A) \supseteq A \quad \textit{and} \quad T_\delta(A) \subseteq A, \quad A \in \Xi.$$

Furthermore, if α is the speed coefficient of T, then $\alpha \wedge 0$ is the speed coefficient of T_β and $\alpha \vee 0$ is the speed coefficient of T_δ. And finally, if T is a shrinker, then $T_\beta(T_\delta)^k$ is a shrinker for all positive integers k.

PROOF. Every conclusion in this result is routine to check, except possibly the last one needs a small hint. The hint is: the speed coefficient of $T_\beta(T_\delta)^k$ is $(\alpha \wedge 0) + k(\alpha \vee 0)$. □

We now turn to the proof of Theorem 18.3.1. Before giving the formal proof, we first sketch the main idea.

The hard part is to show stability when T is a shrinker. One would like to make some sort of direct comparison between the continuous-time process (ξ_t) with rates $(\beta_x, \delta_x + \varepsilon)$, and the discrete-time process (η_t) with Toom operator T and error probability ε. To make such a comparison, the two processes (ξ_t) and (η_t) need to be defined jointly on the same probability space. Then, to get the stability of the process (ξ_t) from the stability of (η_t), it would be sufficient to find some positive integer J such that $\xi_{Jt} \supseteq \eta_t$ for all $t = 0, 1, 2, \ldots$.

This is clearly asking for too much. There are three main problems. The first is that, in continuous time, it is possible for a vacant site x to stay vacant for a long time, even when the birth rate stays large and there are no "errors". Since we want to compare ξ_{Jt} with η_t, this problem is solved by letting J be large, so that if the birth rate at a site x is positive during some time interval $(J(t-1), Jt]$, then a birth at that site is highly likely. The rare "stubborn" vacant sites are included among the errors. With this expanded notion of errors, things can be arranged so that the occurrence of a birth at x at time t in the discrete-time system implies a birth at x during the time interval $(J(t-1), Jt]$ in the continuous-time system.

The second problem is more difficult. It arises from the fact that, in continuous time, it is possible for many related deaths to occur over a short time interval. Such "death-clusters" are unlikely, but arbitrarily large ones do occur in any non-trivial system, and they must be reckoned with. This problem cannot be handled in the same way as the first problem, since the integer J appears in the wrong place in the comparison. We would need to compare $\xi_{t/J}$ with η_t if we wanted to make the probability of a death-cluster small during a single discrete time step.

The solution is to replace T by a different Toom operator, namely $T_\beta(T_\delta)^k$ (see Proposition 18.4.1). We will take k to be proportional to J^2. This "speeds up" the deaths in the discrete-time process, essentially achieving the desired comparison of the preceding paragraph.

The third problem is also due to "death-clusters". If we want to include them among the errors, we need to generalize the types of error sets that we use to construct our discrete-time comparison process, since "death-clusters" involve more than one site. Processes with these kinds of multi-site errors are studied in [BG91]. We will need the estimates in [BG91] to finish off the proof.

We are now ready for the proof of our main result.

PROOF OF THEOREM 18.3.1. We begin with the easy half. Assume that T is an expander. We will first show that there exists a finite set $D \subseteq \mathbb{Z}^d$ such that $T(D^c) \subseteq D^c$. A simple comparison argument shows that we may assume that $\alpha(w) = 0$ for all $w \in W$. By Theorem 18.2.1, T is not an eroder, so there exists a finite set $B \subseteq \mathbb{Z}^d$ such that $T^n(B^c) \neq \mathbb{Z}^d$ for all positive integers n. Let $D_n = T^n(B^c)^c$. Since $\alpha(w) = 0$ for all W, it is easy to see that the sets D^n all lie inside a single bounded subset of $\mathbb{Z}^d$. It follows that there exists a positive integer k such that, for all sufficiently large n, $D_n = D_{n+k}$. Fix such an n, and let $D = D_n \cup D_{n+1} \cup \ldots \cup D_{n+k-1}$. Clearly, D is nonempty and finite. By monotonicity, $T(D^c) = D^c$.

Now we can complete the proof of the easy half. It follows from the monotonicity of T and the definition of Toom rates that $\beta_x(A) = 0$ for all $x \in D$ if $A \subseteq D^c$. Standard properties of interacting particle systems imply that if the perturbed system with rates $(\beta_x, \delta_x + \varepsilon)$ reaches a state in which all of the sites in D are vacant, then all of the sites in D will remain vacant forever after. Since the death rates of a perturbed system are bounded away from 0, it is easy to see that such a system will almost surely reach such a state. Lack of stability at $\mathbb{Z}^d$ is an easy consequence of this fact and translation-invariance.

We turn our attention to the hard part. Assume that T is a shrinker. By the finite range condition, the positive values of the birth rates are bounded away from 0, and the values of the death rates are bounded above. Because of these facts, standard comparison arguments show that it is enough to prove stability at $\mathbb{Z}^d$ in the case for which the birth rates only take the two values 0 and 1 and the death rates only take the two values 0 and M for some $M \geq 1$. We will make these simplifying assumptions throughout the rest of the proof.

It is now quite easy to describe a useful 'graphical' construction of the interacting particle system with rates $(\beta_x, \delta_x + \varepsilon)$. Let

$$B_x, C_x, D_x, \quad x \in \mathbb{Z}^d,$$

be independent Poisson point processes on the time line $[0, \infty)$. For each x, the intensity measures of B_x, C_x, D_x, are respectively λ, $\varepsilon\lambda$, $M\lambda$, where λ is Lebesgue measure on $[0, \infty)$. It can be shown that there exists a Markov process $(\xi_t, t \geq 0)$ with state space Ξ, defined on the same probability space as the Poisson point processes $B_x, C_x, D_x, x \in \mathbb{Z}^d$, such that $\xi_0 = \mathbb{Z}^d$ and for all sites x, $\xi_t(x) = \xi_{t-}(x)$ unless either

$$\begin{aligned} t \in B_x \text{ and } \beta_x(\xi_{t-}) = 1, &\quad \text{in which case } \xi_t(x) = \xi_{t-}(x) \cup \{x\}, \\ \text{or} \quad t \in C_x, &\quad \text{in which case } \xi_t(x) = \xi_{t-}(x) \setminus \{x\}, \\ \text{or} \quad t \in D_x \text{ and } \delta_x(\xi_{t-}) = M, &\quad \text{in which case } \xi_t(x) = \xi_{t-}(x) \setminus \{x\}. \end{aligned}$$

Furthermore, these conditions uniquely determine the process $(\xi_t, t \geq 0)$ almost surely, and this process is an interacting particle system with rates $(\beta_x, \delta_x + \varepsilon)$ and initial state $\mathbb{Z}^d$.

Informally speaking, the conditions say that (i) a birth can occur at a site x only at a time in B_x, and then only if the birth rate at x at time $t-$ is 1, and (ii) a death can occur at a site x only at a time in C_x or D_x, and in the latter case, only if the unperturbed death rate at x at time $t-$ is M. The deaths at x that occur at times in C_x are interpreted as errors at x. They happen at rate ε.

This graphical construction of the perturbed interacting particle system will make it possible for us to define a discrete-time model that can be directly compared with the continuous-time process. Our discrete-time model will be defined with an inductive definition that is similar to (18.1.1), with the error sets E_t being more general than in Section 18.1. These error sets will be determined by the graphical construction described above.

Let J be a positive integer, and let $\varepsilon = 1/J^{250d}$, so that (ξ_t) now depends on J instead of ε. This parameter J will be used to define a discrete-time model (η_t) that will dominate (ξ_t) in a certain sense. The time interval $((t-1)J, tJ]$ of our continuous-time model (ξ_t) will correspond to the t^{th} time step in our discrete-time model (η_t), and we will eventually see that $\eta_t \supseteq \xi_{Jt}$ for $t = 0, 1, 2, \ldots$.

Our discrete-time model will be defined in terms of error sets E_t that are more general than the ones in Section 18.1. These error sets will consist of three types of errors. The first two types are easy to define. For each $t = 1, 2, 3, \ldots$, let

$$E_t^{(1)} = \{x \in \mathbb{Z}^d \colon C_x \cap ((t-1)J, tJ] \neq \emptyset\}$$
$$E_t^{(2)} = \{x \in \mathbb{Z}^d \colon B_x \cap ((t-1)J, tJ] = \emptyset\}.$$

As indicated by the set $E_t^{(1)}$, an error of the first type occurs at x during the t^{th} time step if an ordinary error occurs at x during $((t-1)J, tJ]$ in the continuous-time model. An error of the second type occurs at x during the t^{th} time step if a birth at x during $((t-1)J, tJ]$ is impossible in the continuous-time model because of the fact that $B_x \cap ((t-1)J, tJ] = \emptyset$.

Errors of the third type are slightly more complicated. These errors indicate places in the continuous-time model where a rapid succession of related deaths is possible. Some terminology will be useful. As usual, let r be the range of the Toom operator T. Given an integer $k \geq 2$, a time interval $\mathcal{I}$, and a site $x \in \mathbb{Z}^d$, we say that a sequence $((x_j, t_j), j = 1, 2, \ldots, k)$ is a *k-chain in $\mathcal{I}$ with head* x if (i) $x = x_1$, (ii) $t_1 \in D_x \cap \mathcal{I}$, and (iii) for all $j = 2, \ldots, k$, we have

$$t_j \in D_{x_j} \cap \mathcal{I}, \quad |x_j - x_{j-1}| \leq r, \quad \text{and} \quad t_j > t_{j-1}.$$

The set $\{x_1, \ldots, x_k\}$ is called the *body* of the k-chain.

We now define the third type of error for our discrete-time model. For positive times t and sites x, let

$$E_t^{(3,x)} = \bigcup_{\ell=0}^{J^2-1} \{y \colon y \text{ is in the body of a } 250d\text{-chain}$$
$$\text{in } \left((t-1)J + \tfrac{\ell}{J},\ (t-1)J + \tfrac{\ell+1}{J}\right] \text{ with head } x\},$$

and

$$E_t^{(3)} = \bigcup_{x \in \mathbb{Z}^d} E_t^{(3,x)}.$$

The full error set corresponding to the t^{th} time step is

$$E_t = E_t^{(1)} \cup E_t^{(2)} \cup E_t^{(3)}.$$

Note that the sets $E_1, E_2, E_3, \ldots$ are iid.

We now define a discrete-time Markov process $(\eta_t, t = 0, 1, 2, \ldots)$ inductively as follows. Let $\eta_0 = \mathbb{Z}^d$, and for $t = 0, 1, 2, \ldots$, let

$$\eta_{t+1} = \left(T_\beta\, (T_\delta)^{250dJ^2}\, (\eta_t \setminus E_{t+1})\right) \setminus E_{t+1}.$$

Since T is assumed to be a shrinker, Proposition 18.4.1 implies that the Toom operator $T_\beta(T_\delta)^{250dJ^2}$ is a shrinker for every J.

We claim that for all $t = 0, 1, 2, \ldots$,

$$\xi_{Jt} \supseteq \eta_t. \tag{18.4.1}$$

The case $t = 0$ is obvious. To prove the claim for $t > 1$, we proceed inductively. So assume that $\xi_{Jt} \supseteq \eta_t$, and let x be in the set η_{t+1}. We must show that $x \in \xi_{J(t+1)}$.

Since the definition of η_{t+1} implies that $x \notin E_{t+1}^{(1)} \cup E_{t+1}^{(2)}$, we know that no error occurs at x in the continuous-time system during $(Jt, J(t+1)]$, and we also know that $B_x \cap (Jt, J(t+1)] \neq \emptyset$. Thus, to prove that $x \in \xi_{J(t+1)}$, it is sufficient to show that $\beta_x(\xi_s) = 1$ and $\delta_x(\xi_s) = 0$ for all $s \in (Jt, J(t+1)]$. By the exclusiveness of the birth and death rates, it is enough to show that

$$\beta_x(\xi_s) = 1, \quad \text{for all } s \in (Jt, J(t+1)]. \tag{18.4.2}$$

For the purposes of obtaining a contradiction, assume that $\beta_x(\xi_s) = 0$ for some $s \in (Jt, J(t+1)]$. For $j = 0, 1, \ldots, 250dJ^2$, let

$$\eta^{(j)} = T_\delta^{250dJ^2-j}(\eta_t \setminus E_{t+1}). \tag{18.4.3}$$

By Proposition 18.4.1,

$$x \in \eta_{t+1} \subseteq T_\beta(\eta^{(0)}) \subseteq T_\beta T_\delta(\eta^{(1)}).$$

It follows from the definitions of T_β, T_δ that $\beta_x(\eta^{(1)}) = 1$. Since $\eta^{(1)} \subseteq \eta_t \subseteq \xi_{Jt}$, weak attractiveness implies that a death must occur at some site $y_1 \in \eta^{(1)} \cap (N+x)$ at some time $s_1 \in (Jt, J(t+1)]$, since otherwise we could not have $\beta_x(\xi_s) = 0$ for some $s \in (Jt, J(t+1)]$. Since $\eta^{(1)}$ does not include any sites in E_{t+1}, no error could occur at y_1 at time s_1, so it must be that $\delta_{y_1}(\xi_{s_1^-}) = M$ and $s_1 \in D_{y_1}$.

Now we proceed inductively. Let (y_1, s_1) be as in the preceding paragraph, and suppose that for some $j = 1, \ldots, 250dJ^2 - 1$, there exist space-time points $(y_1, s_1), \ldots, (y_j, s_j)$ such that for $i = 2, \ldots, j$,

$$s_i \in D_{y_i} \cap (Jt, s_{i-1}), \quad \delta_{y_i}(\xi_{s_i^-}) = M, \quad \text{and} \quad y_i \in \eta^{(i)} \cap (y_{i-1} + N).$$

Since $y_j \in \eta^{(j)} = T_\delta(\eta^{(j+1)})$, we know that $\delta_{y_j}(\eta^{(j+1)}) = 0$. As before, it follows that a death must occur at some site $y_{j+1} \in \eta^{(j+1)} \cap (N + y_j)$ at some time $s_{j+1} \in (Jt, s_j)$. Since $\eta^{(j+1)}$ does not include any sites in E_{t+1}, this death cannot be due to an error at y_{j+1}, so $s_{j+1} \in D_{y_{j+1}}$ and $\delta_{y_{j+1}}(\xi_{s_{j+1}^-}) = M$.

Our inductive argument provides us with a sequence

$$(y_1, s_1), \ldots, (y_{250dJ^2}, s_{250dJ^2})$$

having certain properties. These properties imply that the reversed sequence is a $250dJ^2$-chain in $(Jt, J(t+1)]$ whose head is a site in η_t, and none of the sites involved in the chain lie in the set E_{t+1}. Such a chain must contain a $250d$-chain in some interval of the form $(Jt + \ell/J, Jt + (\ell+1)/J]$ for some $\ell = 0, \ldots, J^2 - 1$. Since E_{t+1} contains all of the sites that could be involved in such a chain, we have a contradiction.

We have shown that (18.4.1) holds for all $t = 0, 1, 2, \ldots$. To complete the proof, we will need some estimates that are found in [BG91]. Some work will be required to move from our present context into the context of [BG91].

We will first define a discrete-time process (ζ_t) that is equivalent to a process found in [BG91]. Let $\zeta_0 = \mathbb{Z}^d$, and then define ζ_t inductively by

$$\zeta_{t+1} = T_\beta (T_\delta)^{250dJ^2}(\zeta_t) \setminus E_{t+1,J}, \quad t = 0, 1, 2, \ldots,$$

where $E_{t+1,J}$ denotes the set $\{x + y : x \in E_{t+1},\ y \in B(250dJ^2 r) \cap \mathbb{Z}^d\}$. This differs from the definition of (η_t) in two ways: (i) the error set has been "thickened" by the amount $250dJ^2 r$, and (ii) the error set only appears once in the formula, in contrast to (18.4.3). The thickening of the error set is intended to compensate for the fact that it only appears once in the formula. Indeed, it is easy to check that, since T_δ has range r, the compensation is more than adequate, so $\zeta_t \subseteq \eta_t$ for all times t.

The process (ζ_t) is precisely of the type considered in [BG91], where a comparison is made between discrete-time processes like (ζ_t) and certain auxiliary processes that are denoted in [BG91] by (A_t). The comparison is that $A_t^c \subseteq \zeta_t$, provided the parameters for the process A_t are chosen appropriately. The parameters for (A_t) are denoted in [BG91] as $\alpha, \beta, \varepsilon, \theta$. As we shall see, the appropriate choices for these parameters in our context all depend on J, except for α.

An appropriate choice for α is any positive value of the speed coefficient of the Toom operator $T_\beta (T_\delta)^{250dJ^2}$. The positive part of this speed coefficient equals the speed coefficient of T_β. Since T_β is a shrinker, its speed coefficient does have at least one positive value, and we let α be this value.

An appropriate choice for β is any lower bound on the speed coefficient of $T_\beta (T_\delta)^{250dJ^2}$. This speed coefficient is not less than $-250dJ^2 r$. So we let $\beta = \beta(J) = -250dJ^2 r$.

The parameter ε of the process (A_t) in [BG91] is an error probability. It is sufficient to let it equal

$$P(x \in E_t^{(1)} \cup E_t^{(2)} \cup E_t^{(3,x)}) = 1 - P(x \notin E_t^{(1)})P(x \notin E_t^{(2)})P(E_t^{(3,x)} = \emptyset)$$

(which does not depend on t or x). We denote this quantity by $\varepsilon(J)$.

We need an upper bound on $\varepsilon(J)$. It is easy to check that $P(x \notin E_t^{(1)}) = \exp(-\varepsilon J) \geq \exp(-1/J^{249d})$ and that $P(x \notin E_t^{(2)}) = 1 - \exp(-J)$. To obtain an adequate lower bound on $P(E_t^{(3,x)} = \emptyset)$, we first note that there are $\leq (2r)^{dk}$ choices for the body of a k-chain whose head is at a particular site x. Thus,

$$\begin{aligned} P\left(E_t^{(3,x)} = \emptyset\right) &\geq \left(1 - \sum_{k=250d}^{\infty} P\,(x \text{ is the head of a } k\text{-chain in } (0, 1/J])\right)^{J^2} \\ &\geq \left(1 - \sum_{k=250d}^{\infty} \frac{(2r)^{dk} \exp(-M/J)(M/J)^k}{k!}\right)^{J^2}. \qquad (18.4.4) \end{aligned}$$

A straightforward estimate shows that the right side is bounded below by $(1 - A/J^{250d})^{J^2}$ for some positive constant A. Combining these three bounds, we have

$$\varepsilon(J) \leq 1 - \exp(-1/J^{249d})(1 - \exp(-J))(1 - A/J^{250d})^{J^2}.$$

A further simple estimate gives

$$\lim_{J\to\infty} J^{240d}\, \varepsilon(J) = 0. \tag{18.4.5}$$

Finally, we have the parameter θ from [BG91], which we denote here by $\theta(J)$. Define a random variable X by the formula

$$X = \inf\{c \geq 1 \colon E_1^{(3,\mathbf{0})} \subseteq B(c)\}.$$

The parameter $\theta(J)$ depends on the distribution of X, so it measures, in some sense, the size of our error sets. It is sufficient to let $\theta(J)$ be any positive number such that

$$C(J) = \sup_k \exp\left(\theta(J)(k + 750drJ^2)\right) P(k - 1 < X \leq k) < \infty.$$

The appearance of the term $750drJ^2$ in this expression comes from the "thickening" that was done to our error sets and the fact that the operator $T_\beta(T_\delta)^{250dJ^2}$ has range $250drJ^2$ (see the definitions of μ and Δ in [BG91, Theorem 2]). We will need to choose $\theta(J)$ so that $C(J)$ stays bounded as $J \to \infty$. By making estimates similar to those used for the terms in (18.4.4), we see that $\sup_k \exp(k) P(k-1 < X \leq k)$ stays bounded as $J \to \infty$. So we may take $\theta(J) = 1/(750dJ^2r) = 1/(3|\beta(J)|)$.

With these choices of the parameters α, $\beta(J)$, $\varepsilon(J)$, $\theta(J)$, it is shown in the proof of [BG91, Theorem 2] that $A_t^c \subseteq \zeta_t$, so we have

$$A_t^c \subseteq \xi_{Jt}, \quad t = 0, 1, 2, \ldots.$$

So it is enough to check that the estimates in [BG91] imply that

$$\lim_{J\to\infty} P(\mathbf{0} \in A_t) = 0. \tag{18.4.6}$$

This is precisely the kind of result that appears in the proof of [BG91, Theorem 1]. So we only need to check that the parameters α, $\beta(J)$, $\varepsilon(J)$, $\theta(J)$ defined above satisfy the conditions that appear in the proof of [BG91, Theorem 1].

Here is a brief description of those conditions. The reader will need to refer to the proof of [BG91, Theorem 1] for further details. In the proof of [BG91, Theorem 1], there is a constant K_3 that depends on the parameters α, $\beta(J)$, $\theta(J)$. Since α is constant, and since $\theta(J)$ is proportional to $\beta(J)$, the relationships given in (2-10) in [BG91] imply that, in our context, K_3 grows no faster than $|\beta(J)|^{12d}$. Given our choice of $\beta(J)$, this means that it is sufficient to let K_3 be on the order of J^{24d}. In that same proof, another constant K_2 is chosen, subject to several conditions, the critical one being that $K_2 \geq (K_3)^2$. This means that, in our context, K_2 can be chosen to grow no faster than J^{48d}.

Now we come to the key requirement. Condition (2-12) in the proof of [BG91, Theorem 1] requires that $\varepsilon(J)$ go to 0 faster than $1/(K_2)^5$. This last condition is guaranteed in our context by (18.4.5). For our particular application, all remaining conditions in the proof of Theorem 1 are weaker than the ones just stated. So the estimates given there apply to our situation. In particular, the results labeled (2-21), (2-22), and (2-23) in [BG91] all hold, and these immediately imply our Equation (18.4.6), as desired. □

18.5 Proofs of Lemmas 18.2.1 and 18.2.2

PROOF OF LEMMA 18.2.1. Note that $\mathcal{H}(U, \alpha I) \subseteq \mathcal{H}(R, \alpha I)$, so it is enough to show that if $x \in \mathcal{H}(R, \alpha I)$, then there exists a vector $u \in U$ such that $x \in \mathcal{H}(u, \alpha(u)u)$.

Let x be in $\mathcal{H}(R, \alpha I)$. Then there exists a vector v such that $x \in \mathcal{H}(v, \alpha(v)v)$. If $v \in U$, we are done, so assume $v \notin U$. Thus, $L(v)$ has dimension less than $d-1$. It follows that the hyperplane $\mathcal{P}(v, \mathbf{0})$ can be rotated about the space $L(v)$. Pick a particular $(d-2)$-dimensional 'axis' of rotation that contains the space $L(v)$, and rotate the hyperplane $\mathcal{P}(v, \mathbf{0})$ continuously about that axis. Parameterize the resulting family of rotations with a real parameter s, and let v_s be the result of rotating v with the rotation corresponding to s. We assume that (i) $v_0 = v$; (ii) v_s is a continuous function of s; (iii) the rotations corresponding to positive s are in the opposite direction from the rotations corresponding to negative s; (iv) if $s \in (-1, 1)$, then $\mathcal{P}(v_s, \mathbf{0})$ does not contain any of the sites in $B(2r) \cap \mathbb{Z}^d$ except for those in $L(v)$; (v) both hyperplanes $\mathcal{P}(v_{\pm 1}, \mathbf{0})$ contain a site in $B(2r) \cap \mathbb{Z}^d$ that is not in $L(v)$.

Now consider the family of hyperplanes $\mathcal{P}(v_s, \alpha(v_s)v_s)$, $s \in [-1, 1]$. The definition of α and the fact that T has range r imply that for $s \in (-1, 1)$, these hyperplanes all contain the same sites in $B(r) \cap \mathbb{Z}^d$. Letting $w(v)$ be as in the proof of Proposition 18.2.1, it follows that each of these hyperplanes contains the affine linear space $L(v) + w(v)$. In other words, as the hyperplanes $\mathcal{P}(v_s, \mathbf{0})$ rotate continuously about the space $L(v)$, the corresponding shifted hyperplanes $\mathcal{P}(v_s, \alpha(v_s)v_s)$ rotate continuously about $L(v) + w(v)$. By continuity, the hyperplanes $\mathcal{P}(v(\pm 1), \alpha(v_{\pm 1})v_{\pm 1})$ contain $L(v) + w(v)$. It is easy to see that at least one of the two half-spaces $\mathcal{H}(v_{\pm 1}, \alpha(v_{\pm 1})v_{\pm 1})$ also contains the point x. We may assume that the parameterization has been chosen so that $x \in \mathcal{H}(v_1, \alpha(v_1)v_1)$. Note that our construction ensures that the dimension of $L(v_1)$ is at least one more than the dimension of $L(v)$.

Repeating the above procedure if necessary, we eventually obtain a vector u such that $x \in \mathcal{H}(u, \alpha(u)u)$ and the dimension of $L(u)$ is $d-1$, as desired. □

PROOF OF LEMMA 18.2.2. To streamline notation in this proof somewhat, we will write

$$A(V') = \mathcal{H}(V', \alpha I)^c$$

for any set $V' \subseteq W$. Note that each such $A(V')$ is the intersection of finitely many closed half-spaces in $\mathbb{R}^d$, and hence, is a closed convex set. Our hypothesis is that $A(W) = \emptyset$ and V is a minimal subset of W such that $A(V) = \emptyset$.

Fix a vector $v \in V$ and let $V' = V \setminus \{v\}$. Let X be the set of points $x \in A(V')$ that minimize the quantity

$$d(x) = \inf\{|x - z| : z \in \mathcal{P}(v, \alpha(v)v)\}.$$

Since $A(V') \subseteq \mathcal{H}(v, \alpha(v)v)$, the set X consists of extreme points of $A(V')$, and $d(x)$ is a positive constant for $x \in X$. The minimality of V implies that X is an affine linear subspace of $\mathbb{R}^d$ (possibly consisting of a single point), and X is parallel to $\mathcal{P}(v, \alpha(v)v)$ by construction. Thus, there exists a set $B \subseteq V'$ such that

$$X = \bigcap_{v \in B} \mathcal{P}(v, \alpha(v)v). \tag{18.5.1}$$

Assume that B is the maximal set with this property. We will show that $B = V'$.

Suppose that $B \neq V'$. Since $X \subseteq A(V')$, the maximality of B and the definition of X imply that X lies in the interior of $A(V' \setminus B)$. Thus, for any $x \in X$ and $y \in A(B)$, a part of the line segment from x to y also lies in the interior of $A(V' \setminus B)$. By the choice of x, every point on the line segment connecting y to x lies at least as far from $\mathcal{P}(v, \alpha(v)v)$ as does x itself. Therefore, $\langle y - x, v\rangle \leq 0$ for all such y. Since $X \subseteq \mathcal{H}(v, \alpha(v)v)$, it follows that the ray starting at x and pointing in the direction of $y - x$ is entirely contained in $\mathcal{H}(v, \alpha(v)v)$. Thus, $A(B)$ is entirely contained in $\mathcal{H}(v, \alpha(v)v)$, so that $\mathcal{H}(B \cup \{v\}, \alpha I) = \mathbb{R}^d$. The minimality of V now implies that $B = V'$.

Again let x be a point in X. Since $B = V'$, we have $\langle x - \alpha(v')v', v'\rangle = 0$ for all $v' \in V'$. We also have $\langle x - \alpha(v)v, v\rangle < 0$. So x lies in the closure of each of the sets $\mathcal{H}(w, \alpha(w)w)$, $w \in V$. Each such closure can be written as $\mathcal{H}(-w, \alpha(w)w)^c$. Reflecting through the origin, we find that $-x$ lies in $\mathcal{H}(V, -\alpha I)^c$. It follows that $-X \subseteq \mathcal{H}(V, -\alpha I)^c$.

To complete the proof, it suffices to show that X contains at least one point with rational coordinates. By definition, each vector in V has integer coordinates, and by Proposition 18.2.1, $\alpha(w)$ is rational for each $w \in V$. These facts and (18.5.1) show that X can be expressed as the solution set of a system of linear equations with integer coefficients. The desired conclusion now follows from the method of Gaussian elimination. □

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