ONE - DIMENSIONAL MONOTONIC TESSELATIONS WITH MEMORY

G.A.Galperin
Mechanical and Mathematical Department, Moscow University

Identical finite automata form a two-way infinite one-dimensional system. The state O,I,2,...,n of any automaton at time t depends monotonically on its $r_I + r_2 + I$ neighbors' states at m previous times. A criterion is given to check whether the system blots out any finite perturbation of the state "all automata are zeros". An estimation for the number of operations sufficient to realize this criterion is also given.

The main results of this work are published in |3|, where some of the estimates are less efficient than here. In more detail, but only for m = I, these results are published in |I,2| . This work was inspired by the paper |4| on a similar theme.

I. Formulations

In this paper, x runs through Z and t runs through $Z_+ = I,2,3,\ldots$ We consider the mappings

$$f : Z.Z_+ \to \{O;I;2;\ldots;n\} ,$$

of which $f_t(x)$ are components. Restrictions of f to lines where t is fixed are denoted by f_t and are called <u>states</u> (at the t moments). We denote by $M_t(x)$ the following $m\cdot(r_I+r_2+I)$ matrix, which depends on f:

$$M_t(x) = \left| \begin{array}{cccc} f_{t-I}(x-r_I), & f_{f-I}(x-r_I+I), & \ldots , & f_{t-I}(x+r_2) \\ f_{t-2}(x-r_I), & f_{t-2}(x-r_I+I), & \ldots , & f_{t-2}(x+r_2) \\ \cdots & \cdots & \cdots & \cdots \\ f_{t-m}(x-r_I), & f_{t-m}(x-r_I+I) , & \ldots , & f_{t-m}(x+r_2) \end{array} \right|$$

where m, $r_1 \geq O$, $r_2 \geq O$ are constants and t>m. The value $D = r_I+r_2+I$ will be called a <u>diameter</u>. Let a map $\phi : \{O;I;2;\ldots;n\}^{mD} \to \{O;I;2;\ldots;n\}$ be given, which we shall apply to matrices $M_t(x)$, their elements scanned the standard way. We call a map a <u>tesselation</u>, denoted by P, which transforms an m-tuple of states $f_I, f_2, \ldots, f_m$ into such f, the first m restrictions of which are f_I, $f_2, \dots, f_m$, and the following ones defined successively by the rule:

$$\text{(I)} \qquad \forall x, \quad \forall t > m : f_t(x) = \phi(M_t(x)).$$

In other words, the state $f_t(x)$ of the x-th automaton at time t is a result of the map $\phi(.)$ applied to the states of $x-r_I$, $x-r_I+I$, ... , $x+r_2$-th automata at times $t-I, \dots, t-m$.

We assume, except in specially emphasised cases, that the map $\phi(.)$ is monotonic, that is

$$a_I \le a'_I, \dots, a_{mD} \le a'_{mD} \Rightarrow \phi(a_I, \dots, a_{mD}) \le \phi(a'_I, \dots, a'_{mD})$$

and that $\phi(k,k, \dots, k) = k$ for all $k \in \{O;I;\dots;n\}$.

The basic notions used in this paper are the following. A state f_t is called an island if it has only a finite number of non-zero components $f_t(x) \neq O$. A tesselation P is called a dissolvent if for any islands f_I, f_2, ... , f_m the result f of P's action on $f_I, \dots, f_m$ complies with the following condition:

$$\text{(2)} \qquad \exists T \quad (\forall t > T, \forall x : f_t(x) = O).$$

First let us describe informally the main results of this paper. We introduce rational numbers less than $2 \cdot \binom{n+2}{2}^m$, called velocities, which are essential for P's behavior. THEOREM I infers that velocities can be expressed as fractions, the denominators of which do not exceed mn. THEOREM 2, which is based on it, proves that any velocity can be computed with only $\text{const} \cdot (mn)^4 \cdot D^2$ performances of the map $\phi(.)$. Finally, THEOREM 3 shows the use of velocities. It provides a criterion whether P is a dissolvent in terms of 2n of our velocities. THEOREMS 2 and 3 infer that $\text{const}.m^4.n^5.D^2$ performances of $\phi(.)$ are sufficient to find whether a given P is a dissolvent.

In general, tesselations with any m can be reduced to tesselations with m = I, the number of automata states, or the number of neighbors of each automaton, increasing. But these transformations seem useless in our case because they breach the monotony of $\phi(.)$. It was shown lately by N.V.Petri that there is no algorithm for finding out whether any given P (perhaps, non-monotonic) is a dissolvent. (See |5| , where similar problems are proved to be algorithmically unsolvable even for monotonic P).

Now let us give some definitions. We term the m-tuple of states a <u>hyperstate</u> at a time t and denote it by Φ_t:

$$\Phi_t = (f_{t-m+I},\dots,f_t).$$

We introduce an <u>operator</u> $\mathfrak{P}$, which transforms any m-tuple of states into another m-tuple by the following rule:

$$(3) \qquad \mathfrak{P}\Phi_t = (f_{t-m+2},\dots,f_t,\ f_{t+I}),$$

where components $f_{t+I}(x)$ of f_{t+I} are determined by formula (I); t+I is substituted for t in it.

We introduce a set $S = \{O;I;,..;\ n\}^m$. Elements of S are m-tuples $a = (a_I,\ a_2,\dots,a_m)$, where all $a_k \in \{O;I;\dots;n\}$. Particular elements of S are $\hat{k} = (k,k,\dots,k)$, $O \le k \le n$. S is <u>partially ordered</u> by the rule:

$$(k_I,k_2,\dots,k_m) \prec (k_I',\ k_2',\dots,k_m') \iff k_1 \le k_1',\dots,k_m \le k_m'$$

We write also $a \prec\prec b$ for $a \prec b$ and $a \neq b$.

We shall consider a hyperstate Φ (often omitting the index t) as a sequence of elements of S:

$$\Phi = (\dots,\ \Phi(-2),\ \Phi(-I),\ \Phi(O),\ \Phi(I),\ \Phi(2),\dots);\ \Phi(x) \in S.$$

We shall call $\Phi(x)$ the <u>content</u> of the point $x \in Z$. We shall say that $\Phi \prec \Phi'$ if $\forall x : \Phi(x) \prec \Phi'(x)$.

We define a map $\mathfrak{P}$: S -> S by the rule $\mathfrak{P}(a) = \mathfrak{P}\Phi(x)$, where x is any point and $\Phi(y) = a$ for all y. Our assumption $\phi(k,k,\dots,k) = k$ infers that $\mathfrak{P}(\hat{k}) = \hat{k}$ for all k. Let us imagine an oriented graph $g(\mathfrak{P})$, the sites of which are elements of S. From every $a \in S$ site one oriented bond goes to $\mathfrak{P}(a)$. Of course, $g(\mathfrak{P})$ disintegrates into several <u>basins</u>, each consisting of a <u>cycle</u> and several paths leading to it.

A hyperstate Φ is called <u>increasing</u> if $x<y \Rightarrow \Phi(x) \prec \Phi(y)$ and <u>decreasing</u> if $x<y \Rightarrow \Phi(x) \succ \Phi(y)$. Both increasing and decreasing hyperstates are called <u>stairs</u>. For any Φ stair we call the ordered pair of elements of S

$$(\lim_{x\to-\infty} \Phi(x),\ \lim_{x\to\infty} \Phi(x))$$

its <u>frame</u>, where the meaning of the denotation "lim" is obvious because $\Phi(x)$ becomes constant when $x \to -\infty$ and $x \to \infty$. Of course, if (a,b) is a frame, then $a \prec b$ or $a \succ b$. These cases are symmetric, and we shall confine ourselves to one of them.

Using the monotony of $\phi(.)$, it is easy to show that $\mathcal{P}$ transforms any Φ stair into another stair, the frame of which is determined completely by the frame of Φ. So we can define the action of $\mathcal{P}$ on a frame as follows: $\mathcal{P}(a,b)$ is the frame of $\mathcal{P}\Phi$, where Φ is any stair with the frame (a,b), that is $\mathcal{P}(a,b) = (\mathcal{P}(a), \mathcal{P}(b))$.

We shall use another oriented graph $\bar{g}$, the sites of which are frames, and from every site (a,b) of which an oriented bond leads to $\mathcal{P}(a,b)$. As with elements of S, this graph also consists of basins, each comprising a cycle and paths leading to it.

Let us call an Φ stair constant if $\Phi(x)$ is independent of x. In this connection we call a frame (a,b) constant if a = b. Of course, if one frame in a cycle is constant, so are all the frames in this cycle. We call a frame (a,b) constant-potent if it belongs to a basin, the cycle of which consists of constant frames. For any non-constant stair with a frame of (a,b), the following values pertain:

$$x_L(\Phi) = \min_{\Phi(x)\neq a} x, \qquad x_R(\Phi) = \max_{\Phi(x)\neq b} x \quad .$$

NOTE. If $\Phi \succ \Phi'$ are non-constant stairs of the same frame (a,b), where $a \prec b$, then $x_L(\Phi) \leq x_L(\Phi')$ and $x_R(\Phi) \leq x_R(\Phi')$.

LEMMA I. The following limits exist for any non-constant-potent Φ stair with an (a,b) frame:

$$(4) \qquad L_{a,b} = \lim_{t\to\infty} x_L(\mathcal{P}^t\Phi).t^{-1}, \qquad R_{a,b} = \lim_{t\to\infty} x_R(\mathcal{P}^t\Phi).t^{-1}.$$

These limits are the same for any stair with the given frame (a,b), which warrants our denotations $L_{a,b}$, $R_{a,b}$. Moreover, $L_{a,b}$, $R_{a,b}$ are the same for all (a,b) frames of a basin.

The proof deals with $a \prec b$ only. The well-known facts about limits reduce the problem to a case when (a,b) belongs to a cycle. We denote the length of this cycle by k, that is, $\mathcal{P}^k(a,b) = (a,b)$.

First let us prove that these limits exist for a case when an Φ stair is a jump J (at a point x_0), that is,

$$\Phi(x) = J(x) = \begin{cases} a & \text{if } x \leq x_0 \\ b & \text{if } x > x_0 \;. \end{cases}$$

It is well known that if a numerical sequence A_t (t natural) complies with one of the following restrictions:

$$\forall t,\tau : A_{t+\tau} \le A_t + A_\tau \qquad \text{or} \qquad \forall t,\tau : A_{t+\tau} \ge A_t + A_\tau ,$$

then $\lim_{t\to\infty} A_t t^{-I}$ exists. Now let us fit a $\mathcal{P}^{kt}J$ stair between two jumps at points $x_L(\mathcal{P}^{kt}J)$ and $x_R(\mathcal{P}^{kt}J)$ with the same frame (a,b):

$$\sigma^{x_L(\mathcal{P}^{kt}J)} J > \mathcal{P}^{kt} J > \sigma^{x_R(\mathcal{P}^{kt}J)} J ,$$

where σ^d is a d-translation: $\sigma^d J(x) = J(x-d)$. Now we apply $\mathcal{P}^{k\tau}$ to the three parts of this inequality and permute with the translations:

$$\sigma^{x_L(\mathcal{P}^{kt}J)} \mathcal{P}^{k\tau} J > \mathcal{P}^{k(t+\tau)} J > \sigma^{x_R(\mathcal{P}^{kt}J)} \mathcal{P}^{k\tau} J .$$

Now we can substitute jumps for $\mathcal{P}^{k\tau}J$ in the extreme parts:

$$\sigma^{x_L(\mathcal{P}^{kt}J) + x_L(\mathcal{P}^{k\tau}J)} J > \mathcal{P}^{k(t+\tau)} J > \sigma^{x_R(\mathcal{P}^{kt}J) + x_R(\mathcal{P}^{k\tau}J)} J .$$

This implies

$$x_L(\mathcal{P}^{kt}J) + x_L(\mathcal{P}^{k\tau}J) \le x_L(\mathcal{P}^{k(t+\tau)}J) ,$$

$$x_R(\mathcal{P}^{kt}J) + x_R(\mathcal{P}^{k\tau}J) \ge x_R(\mathcal{P}^{k(t+\tau)}J) ,$$

and proves the existence of the limits

$$\lim_{t\to\infty} x_L(\mathcal{P}^{kt}J)\cdot(kt)^{-I}, \qquad \lim_{t\to\infty} x_R(\mathcal{P}^{kt}J)\cdot(kt)^{-I} .$$

This infers the existence of the limits (4) because

$$x_L(\mathcal{P}^{kt+r}J) - x_L(\mathcal{P}^{kt}J) = O(I),$$

$$x_R(\mathcal{P}^{kt+r}J) - x_R(\mathcal{P}^{kt}J) = O(I)$$

for $I \le r \le k-I$.

Now let us prove that the limits for an Φ stair with the same frame (a,b) exist, and equal those of the jump. Indeed, we can fit any Φ stair between two jumps:

$$\sigma^{x_L(\Phi)} J > \Phi > \sigma^{x_R(\Phi)} J$$

where J is our jump at the point 0 with the same (a,b) frame. Hence,

$$|x_L(\mathcal{G}^t\phi) - x_L(\mathcal{G}^t J)| \leq \text{const} ,$$

$$|x_R(\mathcal{G}^t\phi) - x_R(\mathcal{G}^t J)| \leq \text{const} ,$$

whence the lemma follows immediately.

<u>DEFINITION I</u>. The limits $L_{a,b}$ and $R_{a,b}$ are called <u>left</u> and <u>right</u> (a,b) - <u>velocities</u>.

<u>NOTE</u>. Of course, $L_{a,b} \leq R_{a,b}$. If they are equal, we denote them by $V_{a,b}$.

Now we can formulate our theorems.

THEOREM I. <u>Every $L_{a,b}$ and $R_{a,b}$ velocity is a fraction, the lowest denominator of which does not exceed</u> $\sum_{i=I}^{m} |b_i - a_i|$ <u>where</u> $a=(a_I,\ldots,a_m)$, $b = (b_I,\ldots,b_m)$.

THEOREM 2. <u>Every $L_{a,b}$ and $R_{a,b}$ velocity can be computed with</u> const. $\cdot(\sum_{i=I}^{m} |b_i-a_i|)^4 \cdot D^2$ <u>performances of the</u> $\phi(.)$ <u>map</u>.

THEOREM 3. <u>A tesselation P is a dissolvent if, and only if, the following n inequalities hold</u>:

(5) $$R_{\hat{0},\hat{k}} > L_{\hat{k},\hat{0}} \quad \underline{\text{for all}} \quad k \in \{I;2;\ldots;n\} .$$

2. <u>Proofs</u>.

<u>LEMMA 2</u>. For any $\mathcal{G}$ and any $a,b \in S$ there are such $c,d \in S$ that $L_{a,b}=V_{a,c}$ and $R_{a,b} = V_{d,b}$.

<u>Proof</u> for m=I is given in LEMMA 5 in |I|. If m=I and a<b, one can take c=a+I, d=b-I. In general, c should be sought among those elements of S which differ from a by I in one component. (It is analogous for d). Full proof is left to the reader.

LEMMA 2 reduces proof of THEOREMS I and 2 to frames, only when $L_{a,b} = R_{a,b}$. It infers also the following

<u>COROLLARY</u>. For any stair Φ with a non-constant-potent frame of (a,b)

$$x_R(\mathcal{G}^t\Phi) = R_{a,b}t + O(I),$$

$$x_L(\mathcal{G}^t\Phi) = L_{a,b}t + O(I) .$$

Proof. Lemma 2 reduces it to the case when $R_{a,b} = L_{a,b}$. In this case the assertion can be proved by induction over the value of $\sum_{i=I}^{m} |b_i - a_i|$. (See also Lemma 5 in |I|).

LEMMA 3. Let $L_{a,b} = R_{a,b}$. Then there is a sequence of $\Phi_I, \Phi_2, \ldots, \Phi_l$ stairs with the frame (a,b), where $l \leq \sum_{i=I}^{m} |b_i - a_i|$, such that

$$\mathcal{P}\Phi_I = \sigma^{k_I}\Phi_2, \ldots, \mathcal{P}\Phi_{l-I} = \sigma^{k_{l-I}}\Phi_l, \quad \mathcal{P}\Phi_l = \sigma^{k_l}\Phi_I,$$

where $k_I, k_2, \ldots, k_l$ are integers.

This lemma infers THEOREM I immediately. Indeed, we have $\mathcal{P}^l\Phi_I = \sigma^k\Phi_I$, where $k = \sum_{i=I}^{l} k_i$, whence $V_{a,b} = k/l$. Its proof is based on LEMMAS 4,5 and 6 and will be given below.

Let us immers our discrete Z axis of x's into a real R axis. So x is real now. A generalized Φ_t hyperstate is now a piecewise constant map $\Phi_t : R\cdot \{t-m+I, \ldots, t\} \rightarrow \{0; I; \ldots; n\}$ and $\mathcal{P}$ denotes a new operator acting on a new Φ_t by a rule identical with the old one, in formula (3). We shall often write Φ for Φ_t. Of course, $\Phi(x) \in S$ for all $x \in R$. As before, Φ is an increasing one if $x<y \Rightarrow \Phi(x) \prec \Phi(y)$, and a decreasing one if $x<y \Rightarrow \Phi(x) \succ \Phi(y)$. Both increasing and decreasing Φ's are stairs (R-stairs). Translations σ^d are also generalized for real d: $\sigma^d\Phi(x) = \Phi(x-d)$.

DEFINITION 2. Two Φ and Φ' stairs are called equivalent (this is written as $\Phi \sim \Phi'$) if one of them is a translate of the other:

$$\Phi = \sigma^d\Phi'.$$

Thus, all stairs with a fixed (a,b) frame are divided into equivalence classes, or just classes. A class containing Φ is denoted by (Φ). A $\mathcal{P}$ operator can be considered as acting on classes.

Let us call any maximal interval in R a piece (in given Φ), on which $\Phi(x)$ is constant, called the content of the corresponding piece. Thus, any Φ hyperstate breaks the R axis into pieces. If Φ is a non-constant stair, the number of the pieces is finite and their lengths are limited, except for the two extreme ones. Of course, no two pieces have the same content, and all contents lie between a and b. Therefore, any stair can be characterized by an s-tuple of non-negative numbers (two of which equal ∞) $(\xi_c) \in R_+^s$, where ξ_c is the length of the piece, the content of which is c, where $a \prec c \prec b$ ($\xi_c = 0$ if there is no such piece) and s is the cardinality of the set $S_{a,b} = \{c \in S : a \prec c \prec b\}$. The subset $\{c \in S_{a,b} : \xi_c > 0\}$ is linearly ordered for any stair. Hence, no more than $\sum_{i=I}^{m} |b_i - a_i| + I$ values among ξ_c can be strictly positive. Varied as stairs with a given s-tuple (ξ_c) may be, they are all equivalent. Excluding the two infinite values ξ_a and ξ_b, we obtain an (s-2)-tuple $(\xi_c) \in R_+^{s-2}$.

Let us call an (s-2)-tuple $(\xi_c) \in R_+^{s-2}$ <u>admissible</u> if the set $\{c \in S_{a,b} : \xi_c > 0\}$ is linearly ordered, and denote by $\Lambda_{a,b} = \Lambda$ the set of all admissible (s-2)-tuples. There is a natural one-to-one correspondence between the Λ set and the set of equivalence classes, which we shall also denote by Λ. Now we introduce <u>a metric</u> in Λ by the following rule.

<u>DEFINITION 3</u>. The distance $\rho((\Phi),(\Psi))$ between two classes (Φ), (Ψ) is the least value of $d \geq 0$ for which there are such $\Phi' \in (\Phi)$ and $\Psi' \in (\Psi)$ that $\Phi' \prec \Psi' \prec \sigma^d \Phi'$. Similarly, $\rho(\Phi,\Psi) = \rho((\Phi),(\Psi))$. Let us check whether ρ is a metric.

I). <u>Symmetry</u>. Suppose that $d = \rho((\Phi),(\Psi))$. Then there are such $\Phi' \in (\Phi)$ and $\Psi' \in (\Psi)$ that

$$\Phi' \prec \Psi' \prec \sigma^d \Phi' .$$

Multiplying this by σ^{-d} we get

$$\sigma^{-d}\Psi' \prec \Phi' \prec \Psi' ,$$

which infers that $\rho((\Psi),(\Phi)) \leq d$, q.e.d.

2). <u>Triangle axiom</u>. Let $\rho((\Phi),(\Psi)) = d_I$, $\rho((\Psi),(\Theta)) = d_2$. Hence, there are such Φ, Ψ_I, Ψ_2, Θ in the corresponding classes that

$$\Phi < \Psi_1 < \sigma^{d_I}\Phi \qquad \text{and} \qquad \Psi_2 < \Theta < \sigma^{d_2}\Psi_2 .$$

By that, $\Psi_2 = \sigma^d \Psi_I$. Hence

$$\Phi < \Psi_I < \sigma^{-d_I}\Theta < \sigma^{d_2}\Psi_I < \sigma^{d_I + d_2}\Phi ,$$

which infers that $\rho((\Phi),(\Theta)) \leq d_I + d_2$.

<u>LEMMA 4</u>. Operator $\mathcal{P} : \Lambda \to \Lambda$ satisfies the Lipschitz condition with the constant equal to I:

$$\rho(\mathcal{P}(\Phi), \mathcal{P}(\Psi)) \leq \rho((\Phi),(\Psi)) .$$

<u>LEMMA 5</u>. Operator $\mathcal{P}$: $\Lambda \to \Lambda$ has a fixed point (Φ_ω) : $\mathcal{P}(\Phi_\omega) = (\Phi_\omega)$.

Let us call an Φ stair a <u>Z-stair</u> if

$$[x] = [y] \Rightarrow \Phi(x) = \Phi(y)$$

($[x]$ denotes the largest integer $x' \leq x$).

LEMMA 6. Let us denote by N the number of equivalence classes of those Z-stairs, the distances of which from the fixed point (Φ_ω) are less than I. Then

$$I \leq N \leq \sum_{i=I}^{m} |b_i - a_i| \text{ , where } (a,b) \text{ is a frame of an } \Phi_\omega \text{ stair.}$$

Proof of LEMMA 4. Let $\Phi \prec \Psi \prec \sigma^d \Phi$. Then $\mathcal{P}(\Phi) \prec \mathcal{P}(\Psi) \prec \mathcal{P}(\sigma^d \Phi) = \sigma^d \cdot \mathcal{P}(\Phi)$, q.e.d.

Proof of LEMMA 5. We introduce an operator $Q : \Lambda \to \Lambda$ which transforms a stair with lengths of pieces (ξ_a), $a \in S$, into a stair with corresponding lengths equal to $\min\{\xi_a; D\}$. Geometrically speaking, this means that any point of the cube K with the edge D in our metrical space (Λ,ρ) remains immobile, while any point beyond K moves to the border of K in the shortest way. Now the superposition $Q\mathcal{P}$ is continuous in (Λ,ρ), as both $\mathcal{P}$ and Q are continuous. Besides, $Q\mathcal{P}$ transforms the ball with a center of O and a radius of mnD within itself. Hence $Q\mathcal{P}$ has a fixed point $(\Phi_\omega) = Q\mathcal{P}(\Phi_\omega)$, according to theorem Brouwer's. This class (Φ_ω) is the one we sought. To prove it, we shall prove three lemmas about Φ_ω.

LEMMA 7. No ξ_c length of a piece in Φ_ω exceeds D.

Proof. This is obviously right for any result of the action of Q. It infers that

$$x_R(\Phi_\omega) - x_L(\Phi_\omega) \leq D. \sum_{i=I}^{m} |b_i - a_i| .$$

LEMMA 8. Let us denote by ξ_c^t the lengths of pieces with contents of c in a $\mathcal{P}^t\Phi_\omega$ stair. Every sequence ξ_c^O, ξ_c^I, ξ_c^2,... is an arithmetical progression.

Proof. We know that $\xi_c^O \leq D$ for all c's. First suppose that $\xi_c^O < D$. The length of the piece with the c content in $Q\mathcal{P}\Phi_\omega$ equals ξ_c^O. On the other hand, it equals $\min\{\xi_c^I; D\}$, whence $\xi_c^I = \xi_c^O$. By the same reasoning $\forall t : \xi_c^t = \xi_c^O$. Now let $\xi_c^O = D$. In this case $\xi_c^I = \xi_c^O$ infers $\forall t : \xi_c^t = \xi_c^O$. Otherwise ξ_c^t is an arithmetical progression because of the locality of $\mathcal{P}$.

LEMMA 9. $L_{a,b} = x_L(\mathcal{P}\Phi_\omega) - x_L(\Phi_\omega)$ and

$$R_{a,b} = x_R(\mathcal{P}\Phi_\omega) - x_R(\Phi_\omega).$$

Proof. It follows from LEMMA 8 that the sequences $x_L(\mathcal{P}^t\Phi_\omega)$ and $x_R(\mathcal{P}^t\Phi_\omega)$, t=O,I,2,..., are arithmetical progressions. Hence, the limits

$$\lim_{t\to\infty} x_L(\mathcal{P}^t\Phi_\omega)\cdot t^{-I} \quad \text{and} \quad \lim_{t\to\infty} x_R(\mathcal{P}^t\Phi_\omega)\cdot t^{-I}$$

equal their differences, which proves the lemma.

Now we can prove LEMMA 5. Since we assume that $L_{a,b} = R_{a,b}$, all the differences of the ξ_c^t arithmetical progressions should be zero, whence $\mathcal{P}(\Phi_\omega)$ has the same lengths of pieces as Φ_ω, that is, $\mathcal{P}(\Phi_\omega)$ is a translate of Φ_ω , q.e.d.

NOTE. By the way, we have proved that if J is a jump with an (a,b) frame, then

$$x_R(\mathcal{P}^t J) - x_L(\mathcal{P}^t J) \leq 2D \sum_{i=I}^{m} |b_i - a_i|$$

for all t's.

Indeed, let us denote $d = D.\ \sum_{i=I}^{m} |b_i-a_i|$. We have seen that $x_R(\Phi_\omega)-$ $-X_L(\Phi_\omega) \leq d$. So, assuming $a \prec b$, we can fit J between two translates of Φ_ω : $\Phi_\omega \succ J \succ \sigma^d\Phi_\omega$, whence $\mathcal{P}^t(\Phi_\omega) \succ \mathcal{P}^t(J) \succ \sigma^d\cdot\mathcal{P}^t(\Phi_\omega)$. This infers

$$x_L(\mathcal{P}^t(\Phi_\omega)) \leq x_L(\mathcal{P}^t(J)),$$

$$x_R(\mathcal{P}^t(J)) \leq x_R(\sigma^d\, \mathcal{P}^t(\Phi_\omega)) = x_R(\mathcal{P}^t(\Phi_\omega)) + d,$$

whence

$$x_R(\mathcal{P}^t J) - x_L(\mathcal{P}^t J) \leq x_R(\mathcal{P}^t\Phi_\omega) - x_L(\mathcal{P}^t\ \Phi_\omega) + d \leq 2d.$$

We shall use this note in proving THEOREM 2.

Proof of LEMMA 6. First let us prove that $I \leq N$. Let us fix a stair $\Phi_\omega \in (\Phi_\omega)$ and denote its points of discontinuity by $x_I, x_2, \ldots, x_q$. Now let Φ_I be a Z-stair having $[x_I], \ldots, [x_q]$ as points of discontinuity, and for which the content of $([x_k], [x_{k+I}])$ is the same as the content of (x_k, x_{K+I}) for Φ_ω. Of course, $\rho(\Phi_I, \Phi_\omega) < I$, whence Φ_I is a specimen of the Z-stairs sought.

Now let us prove that $N \leq \sum_{i=I}^{m} |b_i-a_i|$. In the previous part of the proof we could translate Φ_ω , that is, take another element of (Φ_ω) , before constructing Φ_I. Let us calculate how many different classes of Z-stairs we might obtain in this way. We shall follow how the Φ_I changes when built in the way just described, using $\sigma^d\Phi_\omega$ instead of Φ_ω , where d increases continuously from O to I. (Other values of d are redundant, because d and d+I produce equivalent Z-stairs). Of course, Φ_I may change its class only when some one of x_k+d becomes an integer. Therefore the number of non-equivalent Φ_I's obtained in this way does not exceed the number q of the points of discontinuity of Φ_ω. This q does not exceed $\sum_{i=I}^{m} |b_i-a_i|$ because the set of contents of $\Phi_\omega(x)$ is linearly ordered.

Now let us prove that the (Φ_I) classes constructed in this way <u>exhaust</u> all (Φ)'s, such that $\rho((\Phi),(\Phi_\omega))< I$. Indeed, suppose that

$$\sigma^{d_I}\Phi_\omega \preceq \Phi \preceq \sigma^{d_2}\Phi_\omega, \qquad \text{where } d_2-d_I< I.$$

This implies that all the points of discontinuity of $\sigma^{d_2}\Phi_\omega$ differ from those of Φ by less than I, whence $\Phi=\Phi_I$ built in the described way using $\sigma^{d_2}\Phi_\omega$ for Φ_ω.

Thus, LEMMA 6 and THEOREM I are proved.

<u>Proof of THEOREM 2</u>. By virtue of LEMMA 3, we may confine ourselves to the case of $L_{a,b}=R_{a,b}$. Let us describe a method for computing $V_{a,b}$. We take a jump J (at the point O) with this (a,b) frame and apply our $\mathcal{P}$ operator t times to J. We use the fact that

$$(6) \qquad x_R(\mathcal{P}^tJ) - x_L(\mathcal{P}^tJ) \le 2D \cdot \sum_{i=I}^{m} |b_i-a_i|$$

has been proved by LEMMA 6 (see NOTE). On the other hand

$$(7) \qquad x_L(\mathcal{P}^tJ)\cdot t^{-I} \le V_{a,b} \le x_R(\mathcal{P}^tJ)\cdot t^{-I}$$

for any t. So, the larger the t, the better estimate $V_{a,b}$. It is sufficient to take $t = 2D\cdot(\sum_{i=I}^{m} |b_i-a_i|)^3$. Indeed, with this value for t, the length of the gap in (7) does not exceed $(\sum_{i=I}^{m} |b_i-a_i|)^{-2}$. But no segment of this length can contain more than one fraction, the denominator of which not exceeding $\sum_{i=I}^{m} |b_i-a_i|$; only such fractions are possible for values of $V_{a,b}$. So the estimate (7) with the chosen value for t determines $V_{a,b}$ uniquely.

The way to compute $V_{a,b}$ is clear. Estimation of the number of performances of the $\phi(.)$ map is based on (6). Indeed, we need not perform the $\mathcal{P}$ action beyond the segment $[x_L(^t\Phi), x_R(^t\Phi)]$, because we know that the content there is $a \in S$ on the left, and $b \in S$ on the right. Performance of the $\mathcal{P}$ action in this segment requires t times $2D\cdot\sum_{i=I}^{m} |b_i-a_i|$, that is $4D^2\cdot(\sum_{i=I}^{m} |b_i-a_i|)^4$ performances of $\phi(.)$, q.e.d.

<u>LEMMA IO.</u> Let J be an $(\hat{O},\hat{k})$ jump at a point O. Then the length of any piece with an $a\in S$ content, such that $\widehat{k-I} \ll a \ll \hat{k}$, in all $\mathcal{P}^tJ$ hyperstates does not exceed a constant independent of t.

Proof. If there is a t in which $\mathcal{P}^t(a) = \hat{k-I}$, the lemma's assertion is obvious. If there is no such t, the lemma is inferred by COROLLARY of LEMMA 2 and the following inequality:

$$(8) \qquad R_{\hat{O},a} \geq R_{\hat{O},\hat{k}} \quad ,$$

which we are going to prove. Let $a_I, a_2, \ldots, a_M$ form the cycle of the basin containing a. The condition

$$\forall t : \hat{k-I} \lll \mathcal{P}^t(a)$$

which we have just assumed, infers that

$$(9) \qquad \max \{a_I, a_2, \ldots, a_M\} = \hat{k}.$$

Now let J_s be an $(\hat{O}, a_s)$ jump at a point O, where $I \leq s \leq M$. We denote

$$x_R(t) = \max_{I \leq s \leq M} x_R(\mathcal{P}^t J_s) \ .$$

Of course, $x_R(t) - t \cdot R_{\hat{O},a} = o(t)$. On the other hand, $J_s \prec J$ infers that

$$\mathcal{P}^t J_s(x_R(t)) \prec \mathcal{P}^t J(x_R(t))$$

for all $s \in \{I;2;\ldots;M\}, t \in Z_+$. This and (9) infer that $\mathcal{P}^t J(x_R(t)) = \hat{k}$, that is, $x_R(\mathcal{P}^t J) \leq x_R(t)$, whence (8) follows immediately.

Proof of THEOREM 3. First we assume that $R_{\hat{O},\hat{k}} > L_{\hat{k},\hat{O}}$ for all $k \in \{I;\ldots n\}$ and prove that our tesselation is a dissolvent. From monotony, it is sufficient to prove that it "dissolves" any initial state of the form $\Phi = \ldots \hat{O}\ \hat{O}\ \hat{O}\ \hat{n}\ \hat{n}\ \hat{n} \ldots \hat{n}\ \hat{n}\ \hat{n}\ \hat{O}\ \hat{O}\ \hat{O} \ldots$
For this we shall prove by induction reasoning over k from I to n, that it dissolves any state of the form

$$(IO) \qquad \ldots \hat{O}\ \hat{O}\ \hat{O}\ \hat{k}\ \hat{k}\ \hat{k} \ldots \hat{k}\ \hat{k}\ \hat{k}\ \hat{O}\ \hat{O}\ \hat{O} \ldots$$

Suppose that it is already proved for k-I. Now prove it for k. Let x,y be the coordinates of the ends of the piece of $\hat{k}$ -s in (IO). Then we majorize (IO) in two jumps : J_I is a $(\hat{O},\hat{k})$ jump at the point x, and J_2 is a $(\hat{k},\hat{O})$ jump at the point y. Now let us put

$$t = \left[(y - x + 2C) \cdot (R_{\hat{O},\hat{k}} - L_{\hat{k},\hat{O}})^{-I}\right] + I,$$

where C is more than the sum of the constants exceeding (by virtue of LEMMA IO) the lengths of all the pieces in $\mathcal{P}^t J_I$ and $\mathcal{P}^t J_2$ with a contents, where $\hat{k-I} \lll a \lll \hat{k}$. Then the contents of all points in $\mathcal{P}^t \Phi$ do not exceed $\widehat{k-I}$. By induction supposition, $\mathcal{P}^t \Phi$ is dissolved, and so is Φ.

Now let us assume $R_{\hat{O},\hat{k}} \leq L_{\hat{k},\hat{O}}$ for some $k \in \{I;2;\ldots;n\}$. Then an initial Φ state of the form

$$\ldots \hat{O}\ \hat{O}\ \hat{O}\ \hat{k}\ \hat{k}\ \hat{k}\ \ldots\ \hat{k}\ \hat{k}\ \hat{k}\ \hat{O}\ \hat{O}\ \hat{O}\ \ldots$$

is "undissolvable", provided the piece with the $\hat{k}$ content is long enough. In fact, let J_L and J_R be jumps with frames $(\hat{O},\hat{k})$ and $(\hat{k},\hat{O})$:

$$J_L = \ldots \hat{O}\ \hat{O}\ \hat{O}\ \hat{k}\ \hat{k}\ \hat{k}\ \ldots, \quad J_R = \ldots \hat{k}\ \hat{k}\ \hat{k}\ \hat{O}\ \hat{O}\ \hat{O}\ \ldots$$

It follows from the proof of LEMMA 5 that there is such a constant C, that

$$x_R(\mathcal{P}^t J_L) < R_{\hat{O},\hat{k}}\cdot t + C, \quad x_L(\mathcal{P}^t J_R) > L_{\hat{k},\hat{O}}\cdot t - C.$$

(C may be taken as equal to mkD). This fact and the locality of $\mathcal{P}$ infer that $\mathcal{P}^t\Phi$ for any t contains a piece with a k content, provided the length of the piece with the k content for Φ exceeds 2C+D.

3. Notes and Examples

NOTE I. All the proofs can be transformed to fit a more general case when the restriction $\phi(a,a,\ldots,a) = a$ is rejected.

Here is how we can avoid this condition (of course, the $\ldots\hat{O}\ \hat{O}\ \hat{O}\ \ldots$ state remains an invariant). In general case, there are $O<\alpha_I<\alpha_2 \ldots <\alpha_s<n$ that in all $i = I,2,\ldots,s$, the states $(\ldots\alpha_i\alpha_i\alpha_i\ \ldots)$ for all i's are stationary for P, that is, $\phi(\alpha_i,\ldots,\alpha_i) = \alpha_i$, and all other states $(\ldots\beta\ \beta\ \beta\ \ldots)$ transfer to one of the $(\ldots\ \alpha_i\alpha_i\alpha_i\ \ldots)$ states under the action of a degree of P. Note that in the process of such a transfer, the contents of the intermediate states form a monotonic sequence. The set $\{O;I;\ldots;n\}$ thus divides into s+I groups (O), (α_I), $\ldots,(\alpha_s)$, each of which contains a single element α_i for which $(\ldots\alpha_i\alpha_i\alpha_i\ \ldots)$ is a fixed point so that any $(\ldots\beta\ \beta\ \beta\ \ldots)$ state in a corresponding group transfers to the fixed point after several applications of the P operator. Accordingly, we must apply the concept of the

$$^{R}(\alpha_{i_I}\ldots\alpha_{i_m}),(\alpha_{j_m}\ldots\alpha_{j_m}) \text{ and } ^{L}(\alpha_{i_I}\ldots\alpha_{i_m}),(\alpha_{j_I}\ldots\alpha_{j_m})$$

velocities only for the numbers O, $\alpha_I, \alpha_2,\ldots,\alpha_s$. All our three theorems can fit this new case by substituting $\alpha_I, \alpha_2,\ldots,\alpha_s$ for $I,\ldots,n$.

EXAMPLE I. Let us illustrate our considerations by a monotonic tesselation P_I with m = I, n = 2, $r_I = r_2 = I$ (see |I|). The following table defines the map ϕ: $\{O;I;2\}^3 \to \{O;I;2\}$ (the omitted values should be chosen such that $\phi(.)$ is monotonic).

0 0 I ↓ 0	0 I I ↓ I	0 I 2 ↓ I	0 2 2 ↓ I	0 0 2 ↓ 0	I I 2 ↓ I	I 2 2 ↓ I
2 2 I ↓ 2	2 2 0 ↓ 2	2 I I ↓ 2	2 I 0 ↓ 2	2 0 0 ↓ I	I I 0 ↓ 0	1 0 0 ↓ 0
		0 0 0 ↓ 0	I I I ↓ I	2 2 2 ↓ 2		

Table I.

Applying P_I to various jumps, one can see that $V_{I,O} = -I$; $L_{O,2} = V_{O,I} = 0$; $V_{2,O} = I/2$; $R_{O,2} = V_{I,2} = I$; $V_{2,I} = I$.

This is a dissolvent. An island

$$\ldots OOO\underbrace{III\ldots III}_{l_I}\underbrace{222\ldots 222}_{l_2}\underbrace{III\ldots III}_{l_3}OOO\ldots$$

vanishes (becomes "all zeros") in a time $t = 2l_I + 4l_2 + l_3$.

NOTE 2. The behavior of some stairs with $t \to \infty$ was examined in more detail in |I|, but only for m=I. A geometrical procedure was offered in the plane (x,t), which allowed us to describe the evolution of pieces of $\mathfrak{P}^t\Phi$ for all t where Φ is a stair, all pieces of which are fairly long. |I| shows that there are only three possibilities for a piece in Φ in this case : either it vanishes, its length remains limited, or its ends move linearly in t up to an additive constant. It is important that the behavior of a piece in these terms remains unchanged under any variations of the lengths of the initial pieces (provided they are large enough and their order on the line is unchanged). For a general initial state, the behaviour of $\mathfrak{P}^t\Phi$, with $t \to \infty$, depends on the lengths of its pieces. EXAMPLE 2 shows this.

EXAMPLE 2. A P_2 tesselation has m = I, n = 2, $r_I = 5$, $r_2 = 0$. Some of the $\phi(a_I,\ldots,a_6)$ values are given in TABLE 2. (Other values should be chosen such that $\phi(.)$ is monotonic. You can check that is possible).

The action of this tesselation upon initial states with identical contents, but various lengths of pieces, may produce qualitatively different effects. You can see this in FIGURE I.

EXAMPLE 3. There are a monotonic P_3 tesselation and an initial Φ state, such that lengths of some pieces depend on t in a rather sophisticated way. We do not present its table of values because it is too extensive. Let us describe how it functions. The $P_3^t\Phi$ state acts like a shuttle because a standard group of pieces moves to and fro along the x axis. So we shall call the $P_3^t\Phi$ states shuttles. We shall have two kinds of shuttles: to-shuttles

0	0	0	0	0	0	⟼	0
0	0	0	0	0	1	⟼	0
0	0	0	0	1	1	⟼	0
0	0	0	1	1	1	⟼	0
0	0	1	1	1	1	⟼	0
0	1	1	1	1	1	⟼	0
1	1	1	1	1	1	⟼	1
1	1	1	1	1	2	⟼	2
0	0	0	0	0	2	⟼	0
0	0	0	0	2	2	⟼	0
0	0	0	2	2	2	⟼	0
0	0	2	2	2	2	⟼	2
1	1	1	1	1	0	⟼	1
1	1	1	1	0	0	⟼	1
1	1	1	0	0	0	⟼	1
1	1	0	0	0	0	⟼	1
1	0	0	0	0	0	⟼	0
2	2	2	2	2	2	⟼	2
2	2	2	2	2	1	⟼	2
2	2	2	2	1	1	⟼	2
2	2	2	1	1	1	⟼	1

...111 ¦ 00000... ...111 ¦ 11110...	$V_{1,0}=4$
...000 ¦ 22222... ...000 ¦ 00022...	$V_{0,2}=3$
...222 ¦ 11111... ...222 ¦ 22111...	$V_{2,1}=2$
...111 ¦ 2222... ...111 ¦ 2222...	$V_{1,2}=0$
...000 ¦ 111111... ...000 ¦ 000001...	$V_{0,1}=5$

TABLE 2.

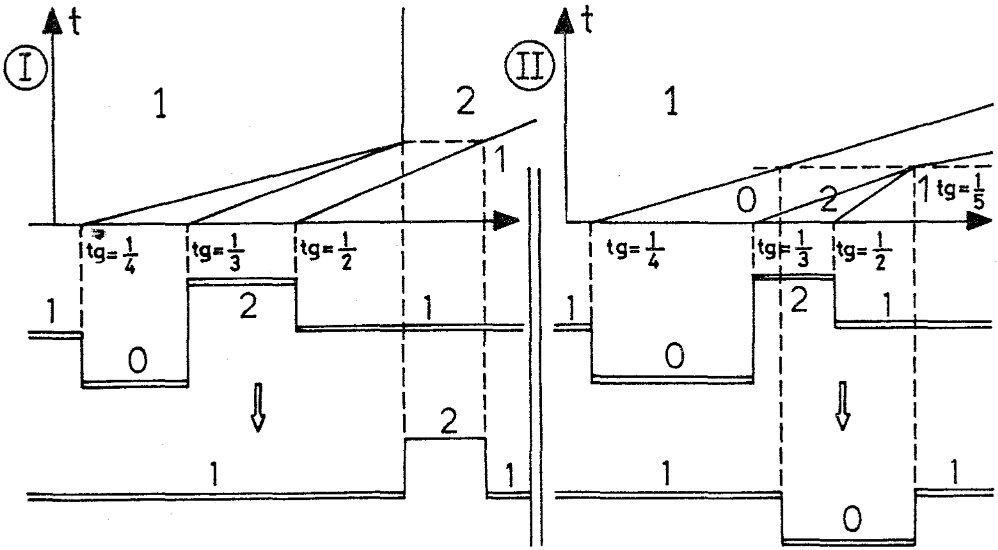

FIGURE I. Action of P_2 upon two initial states with different lengths of pieces.

$$\gamma_{l,r} = \ldots 222\underbrace{\overbrace{333\ldots333}^{l}I504\overbrace{333\ldots333}^{r}}_{d}222\ldots$$

and fro-shuttles

$$\delta_{l,r} = \ldots 222\underbrace{\overbrace{333\ldots333}^{l}405I\overbrace{333\ldots333}^{r}}_{d}222\ldots$$

where d = l+r+4. The action of P_3 upon these shuttles complies with the following rules. The group I504 moves right: $P_3\gamma_{l,r} = \gamma_{l+I,r-I}$, until it reaches the right boundary between the 3's and 2's, and the shuttle becomes $\gamma_{d-5,I}$. Then it bounces back from this boundary, moving it ahead by I at the same time, and turns into 405I. Then it moves left. So a shuttle $\delta_{d-4,I}$ appears. Now the rule $P_3\delta_{l,r} = \delta_{l-I,r+I}$ works. When this group, 405I, reaches the left boundary, a symmetric bounce back occurs, and so on. Thus, if a to-shuttle exist while t_I time-steps, then a t_I+I fro-shuttle exist while t_I time-steps, then a t_I+I fro-shuttle exists, a t_I+2 to-shuttle exists, and so on.

We see that the total length of the two pieces with the 3 content grows as $(2t)^{I/2} + O(I)$, which is not linear.

It is essential that some pieces of all these shuttles be very short. The author can prove that no monotonic tesselation can transform a state all pieces of which are long enough, into a state resembling a shuttle. But there is a non-monotonic tesselation which produces such a shuttle from a state, all pieces of which are arbitrarily large. In fact, these are pieces of a shuttle multiplied by any integer.

In conclusion I thank A.L.Toom for many exciting discussions.

References

I. Galperin G.A. One-dimensional Automata Networks with Monotonic Local Interactions, Problemy Peredaci Informacii I2 (I976) vyp.4.
2. Galperin G.A. On Velocities of Interaction Propagation in One-dimensional Automata Networks, Problemy Peredači Informacii I3 (I977) vyp I.
3. Galperin G.A. One-dimensional Local Monotonic Operators with Memory, DoklAcad.Nauk USSR 228 (I976), 2.
4. Toom A.L. Monotone Binary Tessellation Automata, Problemy Peredaci Informacii I2 (I976), vyp I, 48-54.
5. Toom A.L. and Mityushin L.G. Two Results Concerning Unsolvability for One-dimensional Tessellation Automata, Problemy peredači Informacii I2 (I976), vyp 2, 69-75.