Dokl. Akad. Nauk SSSR
Tom 228 (1976), No. 2

Soviet Math. Dokl.
Vol. 17 (1976), No. 3

ONE-DIMENSIONAL LOCAL MONOTONE OPERATORS WITH MEMORY

UDC 62-507

G. A. GAL'PERIN

1. **Formulations.** Suppose the coordinate x runs through the set $\mathbf{Z}$ of integers and let the coordinate t (time) run through the set $\mathbf{Z}_+$ of natural numbers. We will consider the set of mappings $f: \mathbf{Z} \times \mathbf{Z}_+ \to \{0, 1, \ldots, n\}$. The restrictions of f to the lines $t = 1, 2, 3, \ldots$, will be denoted by f_t and called the *state* at the moment of time t.

We let $M_t(x)$ denote the $(m \times (2r + 1))$-matrix

$$M_t(x) = \begin{pmatrix} f_{t-1}(x-h), & f_{t-1}(x-h+1), \ldots, & f_{t-1}(x+h) \\ f_{t-2}(x-h), & f_{t-2}(x-h+1), \ldots, & f_{t-2}(x+h) \\ \cdots & \cdots & \cdots \\ f_{t-m}(x-h), & f_{t-m}(x-h+1), \ldots, & f_{t-m}(x+h) \end{pmatrix}.$$

Suppose we are given a function ϕ correlating a number $\phi(M_t(x))$ in the set $\{0, 1, \ldots, n\}$ to every matrix $M_t(x)$. The mapping that assigns the mapping f to the states $f_1, f_2, \ldots, f_m$ in such a way that the first m restrictions coincide with $f_1, f_2, \ldots, f_m$, the succeeding f_i being determined inductively by the rule

$$f_t(x) = \varphi(M_t(x)), \quad t > m. \tag{1}$$

will be called the *operator* P.

In other words, the value $f_t(x)$ of the state f_t at the point x is determined by the values of f at mD points ($D = 2r + 1$) arranged in the form of a rectangle over it. When $m = 1$, we obtain a one-dimensional operator with local interaction (mosaic automation).

We will henceforth write the components of $M_t(x)$ in a row (in the natural way). In this article it is assumed that $\phi(a, a, \ldots, a) = a$ for all $a \in \{0, 1, \ldots, n\}$ and that the function ϕ is monotone, i.e. $a_1 \le a'_1, \ldots, a_{mD} \le a'_{mD} \Rightarrow \phi(a_1, \ldots, a_{mD}) \le \phi(a'_1, \ldots, a'_{mD})$.

We will say that f_t is an *island* if only a finite number of the $f_t(x)$, where $x \in \mathbf{Z}$, are nonzero. The operator P will be said to be *eroding* if for arbitrary islands $f_1, f_2, \ldots, f_n$ the state f_t obtained from them by means of P satisfies the condition $\exists T: t > T \Rightarrow f_t \equiv 0$.

We will describe the main results of the work informally. Three theorems are proved here whose statements use $(n + 1)^{2m}$ rational numbers (rates) characterizing P. In Theorem 1 is proved an assertion that implies that the denominators of fractions expressing all the rates are at most mn. This immediately implies Theorem 2 in which the number of operations sufficient for calculating one rate is bounded above

AMS (MOS) subject classifications (1970). Primary 68A25 by const $\cdot (mn)^4 D^2$, a single computation of the function ϕ being considered one operation. Finally, a criterion is given in Theorem 3 that determines for any operator P whether it is eroding. The statement of this criterion involves only $(n+1)^m - 1$ of the $(n+1)^{2m}$ rates introduced; using Theorem 2 we find that at most const $\cdot (mn)^4 D^2 (n+1)^m$ operations must be executed in order to determine whether P is eroding.

We note that the natural reduction of the general case to the case $m = 1$ (by increasing the number of states) does not allow the general case to be investigated, since the new operator turns out to be *nonmonotone*. In fact, for a nonmonotone operator no algorithm can exist to recognize whether the operator is eroding (while some problems turn out to be algorithmically unsolvable even in the monotone case).

Let us pass to the exact statements. We will call the set $f_{t-m+1}, \ldots, f_t$ the *hyperstate* at the moment t:

$$\Phi_t = (f_{t-m+1}, \ldots, f_t). \tag{2}$$

We define an *operator* $\mathcal{P}$ acting on hyperstates by the formula

$$\mathcal{P}\Phi_t = (f_{t-m+2}, \ldots, f_t, f_{t+1}), \tag{3}$$

where $f_{t+1}(x)$ is defined by formula (1), in which t must be replaced by $t+1$.

The hyperstate Φ_t is said to be *monotonically increasing* if $x < y \Rightarrow f_s(x) \leq f_s(y)$ for all s, where s, $t - m + 1 \leq s \leq t$. *Monotonically decreasing* hyperstates are defined analogously. Monotonically increasing and decreasing hyperstates will be called *ladders*.

We introduce the set $S = \{0, 1, \ldots, n\}^m$. The elements of S are vectors of the form $a = (a_1, \ldots, a_m)$, where all the $a_k \in \{0, 1, \ldots, n\}$. We let $\hat{0} = (0, \ldots, 0) \in S$, and $\hat{n} = (n, \ldots, n) \in S$.

We assign to every ladder a type in the form of an ordered pair of elements in S. The type of a ladder is set equal to (a, b), for $a, b \in S$, where $a_k = \lim_{x\to-\infty} f_k(x)$ and $b_k = \lim_{x\to\infty} f_k(x)$, with $t - m + 1 \leq k \leq t$.

Lemma 1. *The operator $\mathcal{P}$ carries every ladder of type (a, b) into a ladder of type (a, b).*

The proof follows from the monotonicity of ϕ.

We also introduce $x_l(\Phi) = \min_{\Phi(x)\neq a} x$ and $x_r(\Phi) = \max_{\Phi(x)\neq b} x$. We will say that a ladder of type (a, b) is an (a, b)*-jump* at the point x_0 if $f_k(x) = (a)_k$ or $(b)_k$, according as $x \leq x_0$ or $x > x_0$.

Let us now introduce the *rates*.

Lemma 2. *For any discontinuity Φ, the following limits exist:*

$$\lim_{t\to\infty} x_l(\mathcal{P}^t\Phi)/t = L_{a,b}, \quad \lim_{t\to\infty} x_r(\mathcal{P}^t\Phi)/t = R_{a,b}. \tag{4}$$

Definition 1. The limits $L_{a,b}$ and $R_{a,b}$ will be said to be the *left* and *right* (a, b)-rates.

Remark. The inequality $L_{a,b} \leq R_{a,b}$ is obvious. In the case of equality these rates will be denoted by $v_{a,b}$.

Theorem 1. *Each of the rates* $L_{a,b}$ *and* $R_{a,b}$ *can be expressed by a fraction whose denominator is at most* $\sum_{i=1}^{m} |b_i - a_i|$ *in modulus.*

Theorem 2. *For each of the rates* $L_{a,b}$ *and* $R_{a,b}$ *there exists a method of calculating it containing at most* const $\cdot (\sum_{i=1}^{m} |b_i - a_i|)^4 D^2$ *reversions to a calculation of* ϕ.

Theorem 3 (criterion). *The operator* P *is eroding if and only if* $(n+1)^m - 1$ *inequalities of the following form are satisfied*:

$$R_{\hat{0}, a} > L_{a, \hat{0}}, \; a \in S, \; a \neq \hat{0}. \tag{5}$$

2. **Proofs.** Let us first discuss the basic points of the proofs. For the case $m = 1$ these proofs have been carried out in detail in two papers I have submitted to the journal *Problemy Peredači Informacii.* We have the following

Lemma 3. *For any* $a, b \in S$, *there exist* $c_1, c_2 \in S$, *such that* $L_{a,b} = v_{a,c_1}$ *and* $R_{a,b} = v_{c_2,b}$.

In proving Theorems 1 and 2 we will therefore consider only ladders of type (a, b) for which $L_{a,b} = R_{a,b} (= v_{a,b})$.

Theorem 1 is obviously implied by

Lemma 4. *Suppose* $L_{a,b} = R_{a,b}$. *Then there exists a set of* (a, b)*-ladders* $\Phi_1, \Phi_2, \ldots, \Phi_l$, *where* $l \leq mn$, *such that* $\mathcal{P}\Phi_1 = \Phi_2$, $\mathcal{P}\Phi_2 = \Phi_3, \ldots, \mathcal{P}\Phi_{l-1} = \Phi_l$, *and* $\mathcal{P}\Phi_l = \Phi_1$.

The proof of this lemma relies on Lemmas 5–7, which we will formulate shortly.

We will consider **Z** (discrete x-axis) as a subset of the real line **R**. The index x runs through all of **R** and serves as a point coordinate. The restriction of the mapping $\Phi: \mathbf{R} \times \mathbf{Z}_+ \to \{0, 1, \ldots, n\}$ to the line t is called the hyperstate Φ_t at the moment t (in the sequel the subscript t on Φ_t will sometimes be omitted), and we extend the action of $\mathcal{P}$ to new hyperstates by a formula literally coinciding with (3). The *operator* σ^d *shifting hyperstates* by $d \in \mathbf{R}$ along the x-axis is defined by $\sigma^d \Phi_t(x) = \Phi_t(x - d)$, where Φ_t is a hyperstate.

Definition 2. The ladders Φ and Ψ will be said to be equivalent ($\Phi \sim \Psi$) if they coincide to within a shift: $\Phi = \sigma^d \Psi$. Thereby the set of all ladders decomposes into equivalence classes $\{(\phi)\}$ (Φ is a representative of the class (Φ)).

We will call $\Phi(x)$ the *level* of the point $x \in \mathbf{R}$, and the maximal segment of points $x \in \mathbf{R}$ of a single level will be called a massive. Clearly, equivalent ladders have identical types (a, b) and lengths of massives of a single level.

A metric ρ can be introduced in the set $\{(\Phi)\}$ of equivalence classes of ladders. Specifically, the distance between two classes (Φ) and (Ψ) is defined as follows. Select ladders $\Phi, \Omega \in (\Phi)$, $\Psi \in (\Psi)$, and a number $d \geq 0$, such that $\Phi > \Psi > \Omega$, $\sigma^d \Phi \leq \Psi$, $\sigma^d \Psi \leq \Omega$, and for any d', where $0 \leq d' \leq d$, at least one of these inequalities is not satisifed. The number d will be called the *distance* between the classes (Φ) and (Ψ) (as well as the distance between any two ladders in these classes). This is written symbolically in the form of a definition.

Definition 3. $\rho(\Phi, \Psi) \leq d \Leftrightarrow (\sigma^d \Phi \leq \Psi, \sigma^d \Psi \leq \Phi)$.

It can easily be verified that ρ is a metric.

In order to formulate the next three lemmas, we will assume that $\mathcal{P}$ operates both on ladders and on equivalence classes of ladders.

Lemma 5. *The operator $\mathcal{P}$ satisfies a Lipschitz condition with constant* 1.

Lemma 6. *The operator $\mathcal{P}$ has the fixed class* $(\Phi)_\omega : \mathcal{P}(\Phi)_\omega = (\Phi)_\omega$.

The restriction of an **R**-ladder to **Z** will be called an *integral ladder* (**Z**-*ladder*). The action of $\mathcal{P}$ on a **Z**-ladder obviously coincides with the restriction to **Z** of the action of $\mathcal{P}$ on the corresponding **R**-ladder.

We also define $\rho(\Phi, \Psi)$, where Φ is a **R**-ladder and Ψ is a **Z**-ladder, by the formula $x \in \mathbf{Z} \Rightarrow (\rho(\Phi(x), \Psi(x)) \leqslant d \Leftrightarrow \sigma^d\Phi(x) \leqslant \Psi(x), \sigma^d\Psi(x) \leqslant \Phi(x))$.

Lemma 7. *The number of equivalence classes of integral ladders that are a distance $\rho \leq 1$ from the fixed class $(\Phi)_\omega$ is not less than one nor greater than* $\Sigma_{i=1}^m |b_i - a_i|$.

The proof of Lemma follows from commutativity with shifts.

To prove Lemma 6 we introduce an operator Q which, in operating on Φ, leaves invariant massive lengths less than $2D$ and assigns the value $2D$ for those massives of length at least $2D$. The operator $Q\mathcal{P}$ is continuous in the metric space $(\{(\Phi)\}, \rho)$ and carries the ball of radius $2nD$ into itself. By the Brouwer theorem, there exists a fixed point $(\Phi)_\omega$ of the operator $Q\mathcal{P}$. Hence we easily see that a representative $\Phi_\omega \in (\Phi)_\omega$ varies under t-fold action of $\mathcal{P}$ in such a way that the lengths of some massives remain invariant while the others increase linearly as $t \to \infty$. Therefore $L_{a,b} = x_l(\mathcal{P}\Phi_\omega) - x_1(\Phi_\omega)$ and $R_{a,b} = x_r(\mathcal{P}\Phi_\omega) - x_r(\Phi_\omega)$. But since we are considering the case when $L_{a,b}$ and $R_{a,b}$ are equal, we find that $(\Phi)_\omega$ is a fixed class for $\mathcal{P}$, and Lemma 6 is proved.

It also follows that a ladder obtained by the t-fold application of $\mathcal{P}$ to a discontinuity Φ of type (a, b) has length (the difference $x_r(\mathcal{P}^t\Phi) - x_1(\mathcal{P}^t\Phi)$) not exceeding $(\Sigma_{i=1}^m |b_i - a_i|) \cdot D$ ($\Sigma_{i=1}^m |b_i - a_i|$ is equal to the maximal possible number of massives for the ladder $\mathcal{P}^t\Phi$; when $a = 0$, $b = \hat{n}$, $\Sigma_{i=1}^m |b_i - a_i| = mn$). This fact will play an important role in proving Theorem 2; meanwhile we pass to the proof of Lemma 7.

The proof of Lemma 7 follows from the fact that every integral ladder at most one unit from $(\Phi)_\omega$ has discontinuities at points in **R** that are integral parts of the discontinuity points of the ladder $\Phi_\omega \in (\Phi)_\omega$, the number of such points of discontinuity being at most $\Sigma_{i=1}^m |b_i - a_i|$.

Proof of Theorem 2. Suppose that $L_{a,b} = R_{a,b} = v_{a,b}$. We consider a discontinuity Φ of type (a, b) at the point 0 and apply the operator $\mathcal{P}$ to it t times. Obviously

$$t^{-1} \cdot x_1(\mathcal{P}^t\Phi) \leqslant v_{a,b} \leqslant t^{-1} \cdot x_r(\mathcal{P}^t\Phi). \tag{6}$$

We select a sufficiently large t (for example, $t = (\Sigma_{i=1}^m |b_i - a_i|)^3 \cdot D$). Then only one fraction with denominator at most $\Sigma_{i=1}^m |b_i - a_i|$ occurs in the interval (6). Thereby $v_{a,b}$ is uniquely defined if the endpoints of (6) are known. It is easy to verify that

$t \cdot (\sum_{i=1}^m |b_i - a_i| \cdot D)$ reversions to the calculation of ϕ are sufficient for finding them. Theorem 2 is proved.

A decisive role in the proof of Theorem 3 is played by the following assertion. We will say that the operator $\mathcal{P}$ is k-eroding, where $k \in \{1, \ldots, n\}$, if for any set of islands $f_1, \ldots, f_m$, the state f obtained from them satisfies the conditions

$$\exists T: t > T \Rightarrow \sum_{\tau=t}^{t+m-1} f_\tau(x) \leq k.$$

Lemma 8. *The operator P is k-eroding if and only if $R_{\hat{0},a} > L_{a,\hat{0}}$ for all $a \in S$ such that $a \neq 0$ and $\sum_{i=1}^m a_i \geq k$.*

All the proofs can be carried over to the case when it is not required that states of the form $(\ldots a, a, \ldots, a, \ldots)$, where $0 < a < n$, be invariant.

In conclusion, I wish to express my appreciation to A. L. Toom, who carefully read through the work and made a number of valuable remarks.

Moscow State University

Received 12/JAN/76

BIBLIOGRAPHY

1. *Homogeneous structures (analysis, synthesis, behavior)*, "Energija", Moscow, 1973. (Russian)

2. A. N. Kolmogorov and S. V. Fomin, *Elements of the theory of function and of functional analysis*, 3rd ed., "Nauka", Moscow, 1972; English transl. of 1st ed., vols. I, II, Graylock Press, Albany, New York, 1957, 1961. MR 19, 44; 22 #9566a.

3. A. L. Toom and L. G. Mitjušin, Problemy Peredači Informacii 12 (1976), no. 2 = Problems of Information Transmission 12 (1976) (to appear).

Translated by R. H. SILVERMAN