RATES OF INTERACTION PROPAGATION IN ONE-DIMENSIONAL AUTOMATON NETWORKS

G. A. Gal'perin

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This paper is a continuation of the author's article "One-dimensional automaton networks with monotonic local interaction." The author determines the values that the left and right rates $L_{a,b}$, $R_{a,b}$ ($0 \le a$, $b \le n$) can assume, and also gives a way of computing them with a complexity corresponding to reality.

1. INTRODUCTION: FORMULATION OF BASIC THEOREMS

Paper [1] investigated the behavior of one-dimensional automaton networks with n + 1 states whose operation is described by a monotonic deterministic operator P. In this paper we will adhere to all the definitions introduced in [1]. The aim of this paper is to determine the values that the left and right rates $L_{a,b}$ and $R_{a,b}$ ($0 \le a$, $b \le n$) can assume, and to give a way of computing them and an estimate for the requisite number of operations involved in doing so.

It follows from [1] that all rates $L_{a,b}$, $R_{a,b}$ are equal to certain rates $v_{a',b'}$, where (a', b') are coupled pairs. The number $v_{a,b}$ is always rational, since there exists a ladder f of type (a, b) for which

$$P^q f = S^p f, \tag{1}$$

where S^1 is the shift-by-1 operator and S^p is the p-th degree of this operator. Number q will be called the period for ladder f.

If, however, (a, b) is not a coupled pair, then [1] implies that there exist left and right rates $L_{a,b}$ and $R_{a,b}$ that are also rational numbers, where $L_{a,b} = v_{a,c}$, $R_{a,b} = v_{d,b}$ for some c and d [a, c, d, b is a monotonic sequence, while (a, c) and (d, b) are coupled pairs]. Therefore to determine which fractions can be used to express the numbers $L_{a,b}$ and $R_{a,b}$, it suffices to consider the case in which (a, b) is a coupled pair and to determine possible values of $v_{a,b}$.

Without loss of generality, we will assume that $a = 0$, $b = n$; pair (0, n) is coupled. The basic results of this paper are Theorems 1 and 2 that follow.

THEOREM 1. The number $v_{0,n}$ can be expressed by a fraction whose denominator does not exceed n (if $v_{0,n} = p/q < 0$, we assume that $p < 0$, $q > 0$).

THEOREM 2. There exists a way of computing $v_{0,n}$ that contains $\asymp D^2 n^4$ operations (D being the diameter of a neighborhood of zero), one-shot computation of the function φ being regarded as one action.*

*The symbol $\asymp$ means "to within a multiplicative constant."

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1. R-Ladders

For what follows, it will be necessary to extend the action of operator P to the entire straight line R. For this we incorporate lattice Z into R as a subset.

We will restrict ourselves to the action of P on ladders. We introduce the notion of an R-ladder.

Definition 1. a) We will say that an R-ladder is specified on R if $f: R \to \{0, 1, \ldots, n\}$ is a monotonic function that is not a constant. Of course, any such function is piecewise-constant.

b) R-ladder f will be called an R-ladder of type (a, b) if $\inf_{x \in R} f(x) = a$, $\sup_{x \in R} f(x) = b$, while numbers a and b will be called the lower and upper levels of the ladder.

Definition b) is similar for $a > b$.

c) For R-ladder f we write

$$x_l(f) = \sup\{x \in R:\ f(x) = a\},$$
$$x_r(f) = \inf\{x \in R:\ f(x) = b\}, \tag{2}$$

and the number $d(f) = x_r(f) - x_l(f)$ will be called the length of ladder f. Obviously, $d(f) \geq 0$.

d) R-ladder f will be called an R-jump if $d(f) = 0$.

Definition 2. Operator $\mathscr{P} = P_R$ that acts on the set of all R-ladders and carries this set into itself is defined by the same natural number r and the same function $\varphi: \{0, 1, \ldots, n\}^{2r+1} \to \{0, 1, \ldots, n\}$ as operator P is; if f is an R-ladder, then for all $x \in R$ we assume that

$$(P_R f)(x) = \varphi(f(x-r), \ldots, f(x+r)); \tag{3}$$

as for operator P, $\varphi(0, \ldots, 0) = 0$ (clearly, $\mathscr{P}$ is a monotonic operator).

It is easy to see that R-ladders considered only on Z yield ladders in the earlier sense. We will now call them Z-ladders (Z-ladders are acted on by operator P – the restriction onto Z of operator $\mathscr{P}$).

Definition 3. Assume that f is a Z-ladder. We will denote by the same letter f the R-ladder defined as follows: If for Z-ladder f for some $k \in Z$ we have $f(k) = f(k+1) = \ldots = f(k+l) = m$, $f(k-1) \neq m$, $f(k+l+1) \neq m$, then for R-ladder f we have

$$f(x) = \begin{cases} m & \text{for} \quad k-1 \leqslant x < k+l+1, \quad k, l \in \mathbf{Z}, \\ \neq m & \text{for} \quad x < k-1 \quad \text{and for} \quad x \geqslant k+l+1. \end{cases} \tag{4}$$

Here the points of discontinuity for the resultant R-ladder f are integer-valued. Any R-ladder for which all discontinuity points are integer-valued will be called an integer-valued ladder.

Definition 4. Segment $K \subset R$ (finite), for all of whose points x we have $f(x) = a$ and for $x \notin K, f(x) \neq a$, will be called a file of level a of R-ladder f, while number f(x) will be called the level of the point $x \in R$.

Remark 1. In what follows we will consider natural exponents of operator $\mathscr{P}$ and arbitrary real exponents of shift-by-1 operator S^1.

Remark 2. R-ladders f_1 and f_2 will be regarded as equivalent if they coincide to within shift; i.e., $f_2 = S^{\alpha} f_1$ for some $\alpha \in R$ (thus the file lengths of one level are the same for these ladders). The set of all R-ladders can be partitioned into equivalence classes $\{(f)\}$: Each class (f) is uniquely defined by any representative of it f.

2. Metric Space (R_+^{n-1}, ρ)

Everywhere in what follows, we will consider R-ladders of type (0, n), and this will not be stipulated specially.

Consider an arbitrary R-ladder f, and assume that $x_1 \leq x_2 \leq \ldots \leq x_n$ are its discontinuity points. The numbers $\Delta_1 = x_2 - x_1$, $\Delta_2 = x_3 - x_2, \ldots, \Delta_{n-1} = x_n - x_{n-1}$; i.e., the lengths of files of f are uniquely defined by the equivalence class (f) to which ladder f belongs (unlike the numbers $x_1, \ldots, x_n$ that depend on the specific representative f). Since $\Delta_i \geq 0$ for all $i = 1, 2, \ldots, n-1$, the vector $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_{n-1}) \in R^{n-1}$ lies in a closed positive octant of space R^{n-1}. A closed positive octant will be denoted by R_+^{n-1}.

Thus there is a one-to-one correspondence between classes $\{(f)\}$ of equivalent R-ladders and points of octant R_+^{n-1}: Point $\Delta = (\Delta_1, \ldots, \Delta_{n-1}) \in R_+^{n-1}$ corresponds to class (f) of R-ladders whose file lengths are $\Delta_1, \ldots, \Delta_{n-1}$.

Consider two equivalence classes (f) and (f'). Obviously, there exist two R-ladders $f \in (f)$, $f' \in (f')$ such that the inequality

$$S^{\lambda_1}f \geqslant f' \geqslant S^{\lambda_2}f \tag{5}$$

holds for them for $\lambda_1, \lambda_2 \in R$. From among all numbers $\{\lambda_1\}$ for which inequality (5) holds, we take the largest one and denote it by $\lambda_1^{(0)}$, while from among all numbers $\{\lambda_2\}$ for which (5) holds, we take the smallest one and denote it by $\lambda_2^{(0)}$ $(\lambda_2^{(0)} \geq \lambda_1^{(0)})$. Assume that classes (f) and (f') correspond to vectors Δ and Δ' of octant R_+^{n-1}.

Definition 5. The distance between vectors Δ and $\Delta' \in R_+^{n-1}$ is the number

$$\rho(\Delta, \Delta') = \lambda_2^{(0)} - \lambda_1^{(0)}. \tag{6}$$

We will demonstrate the correctness of Definition 5; i.e., we will show that function ρ satisfies all the properties of distance. Assume that $x_1', \ldots, x_n'$ are discontinuity points of function f', while $x_1, \ldots, x_n$ are discontinuity points of $S^{\lambda_1^{(0)}}f$. Then it is clear that

$$\rho(\Delta, \Delta') = \max_{1\leqslant i\leqslant n} (x_i' - x_i) - \min_{1\leqslant j\leqslant n} (x_j' - x_j).$$

Let us express $\rho(\Delta, \Delta')$ in terms of the coordinates of vectors Δ and Δ'. Since $x_i' = x_1' + \Delta_1' + \Delta_2' + \ldots + \Delta_i'$, $x_i = x_1 + \Delta_1 + \ldots + \Delta_i$, we have

$$\rho(\Delta, \Delta') = \max_{1\leqslant i\leqslant n} \left[(x_1' - x_1) + \sum_{k=1}^{i} (\Delta_k' - \Delta_k) \right] - \min_{1\leqslant j\leqslant n} \left[(x_1' - x_1) + \sum_{s=1}^{j} (\Delta_s' - \Delta_s) \right]$$

$$= \max_{1\leqslant i\leqslant n} \left[\sum_{k=1}^{i} (\Delta_k' - \Delta_k) \right] - \min_{1\leqslant j\leqslant n} \left[\sum_{s=1}^{j} (\Delta_s' - \Delta_s) \right].$$

We write $\delta_k = \Delta_k' - \Delta_k$; then

$$\rho(\Delta, \Delta') = \max_{1\leqslant i\leqslant n} (\delta_1 + \ldots + \delta_i) - \min_{1\leqslant j\leqslant n} (\delta_1 + \ldots + \delta_j) = \max_{1\leqslant i,\, j\leqslant n} |\delta_{\min(i,j)+1} + \delta_{\min(i,j)+2} + \ldots + \delta_{\max(i,j)}|. \tag{7}$$

We can see that the right side of (7) satisfies all the properties of distance; thus Definition 5 is correct.

Formula (7) also yields that the following lemma is valid.

LEMMA 1. Pair (R_+^{n-1}, ρ), where function ρ is defined by formula (6) [or the equivalent formula (7)], forms a metric space.

Remark 3. The null vector corresponds to a jump of type (0, n).

Remark 4. The length of vector $\Delta = (\Delta_1, \ldots, \Delta_{n-1})$ is the quantity $|\Delta| = \rho(\Delta, 0) = \Delta_1 + \Delta_2 + \ldots + \Delta_{n-1}$.

3. Lipschitz Condition

Operator $\mathscr{P}$ acting on the set of R-ladders induces operator $\mathscr{P}_*$ acting in metric space (R_+^{n-1}, ρ).

LEMMA 2. Mapping $\mathscr{P}_*$ satisfies a Lipschitz condition with constant 1.

Proof. We must prove that for any two vectors Δ, $\Delta' \in R_+^{n-1}$ we have the inequality

$$\rho(\mathscr{P}_*\Delta, \mathscr{P}_*\Delta') \leqslant \rho(\Delta, \Delta'). \tag{8}$$

Assume that (f) and (f') are two different equivalence classes corresponding to vectors Δ and Δ'. We choose representatives of these classes $f \in (f)$, $f' \in (f')$ such that the coordinate of the leftmost point of discontinuity of f is equal to $0 \in R$ and $f \geq f'$. Then Definition 5 yields that if $f \geq f' \geq S^A f$, we have

$$A \geqslant \rho(\Delta, \Delta'). \tag{9}$$

Let $\rho(\Delta, \Delta') = \alpha$; then $f \leq f' \leq S^{\alpha} f$, and in view of the fact that operator $\mathscr{P}$ is monotonic and uniform,

$$\mathscr{P}f \geqslant \mathscr{P}f' \geqslant S^{\alpha}\mathscr{P}f.$$

Inequality (9) and this last expression yield that

$$\alpha \geqslant \rho(\mathscr{P}_*\Delta, \mathscr{P}_*\Delta').$$

The lemma has thus been proved.

Remark 5. Operator $\mathscr{P}_*$ is continuous, since it satisfies a Lipschitz condition.

4. Proof of Theorem 1

Assume that f is an **R**-ladder of type (0, n) and d(f) = N, where N is a number that exceeds the coupling constant for f. Then there exists a natural number T such that

$$d(\mathscr{P}^T f) < N-1. \tag{10}$$

If ladder f in R_+^{n-1} corresponds to vector $\Delta = (\Delta_1, \ldots, \Delta_{n-1})$, then in view of Remark 4 it belongs to set $\mathcal{O} = \{\Delta \in R_+^{n-1} : |\Delta| < N\}$, which is an (n − 1)-dimensional tetrahedron with vertex at 0, the lengths of its edges that depart from 0 being equal to N. It follows from (10) that

$$\mathscr{P}_*^T \mathcal{O} \subset \mathcal{O}. \tag{11}$$

Since $\mathscr{P}_*$ is a continuous operator (see Remark 5), and $\mathcal{O}$ is a convex set, we have from (11) and Brauer's stationary-point theorem [2] that a stationary point σ exists for operator $\mathscr{P}_*^T$ in $\mathcal{O}$:

$$\mathscr{P}_*^T \sigma = \sigma. \tag{12}$$

However, an even stronger assertion is also valid.

LEMMA 3. There exists a point $\xi \in \mathcal{O}$ that is stationary for operator $\mathscr{Q}_*$:

$$\mathscr{P}_* \xi = \xi. \tag{13}$$

Proof. Consider points $\sigma, \mathscr{P}_*\sigma, \mathscr{P}_*^2\sigma, \ldots, \mathscr{P}_*^{T-1}\sigma$ and balls of radius 2N whose centers are at these points:

$$B_{2N}(\sigma), \quad B_{2N}(\mathscr{P}_*\sigma), \ldots, B_{2N}(\mathscr{P}_*^{T-1}\sigma).$$

We denote their intersection with set $\mathfrak{A}$ by $\mathcal{O}$:

$$\mathfrak{A} = B_{2N}(\sigma) \cap B_{2N}(\mathscr{P}_*\sigma) \cap \ldots \cap B_{2N}(\mathscr{P}_*^{T-1}\sigma) \cap \mathcal{O}.$$

1. We have the inclusion $\mathscr{P}_*\mathfrak{A} \subset \mathfrak{A}$.

Indeed, if $x \in \mathfrak{A}$, then $\rho(x, \mathscr{P}_*^k\sigma) \leqslant 2N$ $(0 \le k \le T-1)$ and the Lipschitz condition entails the inequality $\rho(\mathscr{P}_*x, \mathscr{P}_*^{k+1}\sigma) \leqslant 2N$, from which we have $\mathscr{P}_*x \in \mathfrak{A}$. QED.

2. $\mathfrak{A}$ is a convex set, since $B_{2N}(\mathscr{P}_*^k\sigma)$ is convex for all k = 0, 1, . . . , T − 1.

3. $\mathfrak{A} \neq \phi$, since $0 \in \mathfrak{A}$.

Items 1-3 and Brauer's theorem imply that there exists a point $\xi \in \mathcal{O}$, where $\mathscr{P}_*\xi = \xi$. This completes the proof of Lemma 3.

Note that vector ξ corresponds to an equivalence class (f_ξ) such that operator $\mathscr{P}$ acts on any ladder $f_\xi \in (f_\xi)$ like some degree of a shift operator. Ladder f will be called intrinsic.

Lemma 4 which follows will play an important part in completing the proof of Theorem 1.

LEMMA 4. There exist(s) not less than one and not more than n classes of equivalent integer-valued ladders $(f_1), \ldots, (f_l)$ $(l \le n)$ to which there correspond points $\Delta_1, \Delta_2, \ldots, \Delta_l \in R_+^{n-1}$ such that $\rho(\xi, \Delta_i) < 1$ for all $i \le l$ (ξ is a stationary point of operator $\mathscr{P}_*$).

Proof. a) We fix some **R**-ladder $f_\xi \in (f_\xi)$; assume that $x_1, x_2, \ldots, x_n$ are its discontinuity points. On a straight line we mark off the points $[x_1], [x_2], \ldots, [x_n] \in Z$ ([·] is the integer part of the number) and set up an integer-valued ladder f_1 with discontinuities at these points. Z-ladder f_1 belongs to equivalence class (f_1), corresponding to vector Δ_1, where $\rho(\xi, \Delta_1) < 1$. Thus the set of equivalence classes satisfying the conditions of the lemma is not empty.

b) We will show that an arbitrary class (f) of equivalent integer-valued ladders corresponding to vector $\Delta \in R_+^{n-1}$ such that $\rho(\xi, \Delta) < 1$ can be set up by the procedure described in item a).

We take ladder f ∈ (f) whose discontinuity points $y_1, \ldots, y_n$ are integer-valued. Since $\rho(\xi, \Delta) < 1$, there exist ladders $f'_\xi, f''_\xi \in (f_\xi)$ such that $f''_\xi \ge f \ge f'_\xi$ and the discontinuity points of f'_ξ can be obtained by shifting the discontinuity points of f''_ξ by a number $\rho(\xi, \Delta) < 1$ (see Definition 5). If $x_1, x_2, \ldots, x_n$ are discontinuity points of f'_ξ, then for all i, $1 \le i \le n$, $0 \le x_i - y_i \le 1$. But $y_i \in \mathbf{Z}$, and therefore $y_i = [x_i]$ for all $1 \le i \le n$.

c) We will prove that there are not more than n of the desired equivalence classes. Class (f_1) was already set up in a).

We arrange the fractional parts of numbers $x_1, \ldots, x_n$ [discontinuity points of the ladder $f_\xi \in (f_\xi)$ selected in item a)] in increasing order: $\{x_{i_1}\} \le \{x_{i_2}\} \le \ldots \le \{x_{i_n}\}$, and right $\alpha_k = \{x_{i_k}\}$ $(k = 1, \ldots, n)$.

If we perform the procedure of item a) for setting up an integer-valued ladder using **R**-ladder $S^{\lambda_0} f_\xi \in (f_\xi)$ as a starting-point, when $\lambda_0 \le \alpha_1$, then we will clearly obtain a **Z**-ladder of class (f_1) as a result.

If, however, we use **R**-ladder $S^{\lambda_1} f_\xi \in (f_\xi)$ as a starting-point, where $\alpha_1 < \lambda_1 \le \alpha_2$, then the resultant integer-valued ladder f_2, whose discontinuity points are $[x_1 - \lambda_1]$, $[x_2 - \lambda_1]$, $[x_3 - \lambda_1], \ldots, [x_n - \lambda_1]$, will in general belong to equivalence class (f_2), which is not the same as (f_2). Here class (f_2) corresponds to vector $\Delta_2 \in R_+^{n-1}$ satisfying the condition $\rho(\xi, \Delta_2) < 1$.

We can similarly set up integer-valued ladders $f_3 \in (f_3)$ (ladder f_3 had discontinuities of points $[x_1 - \lambda_2]$, $[x_2 - \lambda_2], \ldots, [x_n - \lambda_2]$, where $\alpha_2 < \lambda_2 \le \alpha_3$), $f_4 \in (f_4)$ (the discontinuity points for f_4 being $[x_1 - \lambda_3]$, $[x_2 - \lambda_3], \ldots, [x_n - \lambda_3]$, where $\alpha_3 < \lambda_3 \le \alpha_4$), etc., where equivalence classes (f_3), (f_4), $(f_5)\ldots$ correspond in R_+^{n-1} to vectors Δ_3, Δ_4, $\Delta_5, \ldots$ such that $\rho(\xi, \Delta_3) < 1$, $\rho(\xi, \Delta_4) < 1$, $\rho(\xi, \Delta_5) < 1$, and so forth. But there are not more than n different sets of numbers $[x_1 - \lambda_i], \ldots, [x_n - \lambda_i]$, where $0 \le \lambda_i \le 1$, $\alpha_i < \lambda_i \le \alpha_{i+1}$, $i = 1, \ldots, n-1$, and therefore there are also not more than n different equivalence classes (f_1), $(f_2), \ldots, (f_l)$ nor, correspondingly, vectors $\Delta_1, \ldots, \Delta_l$. Here $\rho(\xi, \Delta_i) < 1$ for all $i = 1, \ldots, l$.

It follows from item b) that an arbitrary equivalence class (f) corresponding to vector Δ, $\rho(\xi, \Delta) < 1$, coincides with one of the already constructed classes (f_i). This exhausts the set of such classes. We see that it consists of $l \le n$ elements. Lemma 4 has thus been proved.

Proof of Theorem 1. In addition to stationary point ξ of operator $\mathscr{P}_*$ in open ball $B_1(\xi)$ of radius 1 with center at ξ there can be integer-valued points $\Delta_1, \Delta_2, \ldots, \Delta_l$ of octant R_+^{n-1}. On the basis of Lemma 4, we have $l \le n$. Since $\rho(\xi, \Delta_i) < 1$ $(i = 1, \ldots, l)$, we have from Lemma 2 that $\rho(\mathscr{P}_*\xi, \mathscr{P}_*\Delta_i) < 1$, i.e., $\rho(\mathscr{P}_*\Delta_i, \xi) < 1$. This yields that $\mathscr{P}_*\Delta_i \in B_1(\xi)$. Since operator $\mathscr{P}$ on the set of integer-valued ladders acts in the same way as operator P on the set of Z-ladders – in particular, it again carries them into integer-valued ladders, we see that $\mathscr{P}_*\Delta_i$ is an integer-valued vector $(i = 1, \ldots, l)$. Consequently, operator $\mathscr{P}_*$ carries set $\{\Delta_1, \ldots, \Delta_l\}$ into itself and we can select a nonempty subset $\tau = \{\Delta_{i_1}, \ldots, \Delta_{i_m}\}$, $m \le l$, such that $\mathscr{P}_*\tau = \tau$, where $\mathscr{P}_*\Delta_{i_1} = \Delta_{i_2}, \ldots$, $\mathscr{P}_*\Delta_{i_m} = \Delta_{i_1}$ (it is natural to call τ the orbit of point ξ). From this we have $\mathscr{P}_*^m \Delta_{i_1} = \Delta_{i_1}$, and if point Δ_{i_1} corresponds to equivalence class (f_{i_1}), then the period for an arbitrary ladder $f_{i_1} \in (f_{i_1})$ is m, $m \le n$. This completes the proof of Theorem 1.

Remark 6. Point ξ can have more than one orbit.

3. PROOF OF THEOREM 2

1. Preliminary Lemma

LEMMA 5. If f is a **Z**-jump of type (0, n), then for all $t \ge 0$ we have $d(P^t f) \le nD$.

Proof. It follows from Lemma 3 that for some real β and an arbitrary **R**-ladder $f \in (f_\xi)$ (ξ being a stationary point of $\mathscr{P}_*$) we have

$$\mathscr{P} f_\xi = S^\beta f_\xi. \tag{14}$$

We will show that

$$d(f_\xi) \le nD. \tag{15}$$

Assume the contrary; let $d(f_\xi) > nD$. Then for **R**-ladder f_ξ the length of some file $[x_1, x_2]$ of level a is strictly greater than D: $x_2 - x_1 > D$. Note that point x_i does not lie in a neighborhood (of diameter D) of point x_j ($i = 1, 2$; $j = 2, 1$; $i \ne j$), and therefore the coordinates of points x_1 and x_2 vary independently under the action $\mathscr{P}$. Therefore if we consider ladder $\hat{f}_\xi$ for which all files except the level-a file are the same as those of f_ξ, while the length of the level-a file is equal to kD, where k is an arbitrary natural number, we will have

$$\mathscr{P} \hat{f}_\xi = S^\beta \hat{f}_\xi. \tag{16}$$

Consequently, for sufficiently large k and all t the number $d(\mathscr{P}^t \hat{f}_\xi)$ will exceed the coupling constant for the pair (0, n), and therefore (0, n) is an uncoupled pair. Thus we arrive at a contradiction.

Hence inequality (15) holds. For Z-jump f we can choose an R-ladder $f_\xi \in (f_\xi)$ such that we have

$$f_\xi > f > S^{d(f_\xi)} f_\xi. \tag{17}$$

Then the monotonicity of $\mathcal{P}$ yields that for all $t \geqslant 0$, $\mathcal{P}^t f_\xi > P^t f > S^{d(f_\xi)} \mathcal{P}^t f_\xi$.

Taking account of (14), we obtain $S^{t\beta} f_\xi > P^t f > S^{t\beta + d(f_\xi)} f_\xi$, from which $d(P^t f) \le (t\beta + d(f_\xi)) - t\beta = d(f_\xi) \le nD$. QED.

2. Procedure for Computing Rate $v_{0,n}$

Consider a jump f of type (0, n) at zero. We apply operator P to it n^3D times and determine the left and right coordinates of ladder $P^{n^3D}f$. We write

$$x_l = x_l(P^{n^3D} f), \quad x_r = x_r(P^{n^3D} f).$$

Then we take a fraction p / q satisfying the following two inequalities:

$$q \leqslant n \tag{18}$$

(if $p/q < 0$, we set $p < 0$, $q > 0$),

$$x_l / n^3D \leqslant p/q \leqslant x_r / n^3D. \tag{19}$$

We assert that

(α): this fraction is unique and

(β): $v_{0,n} = p/q$.

3. Proof of Assertions (α) and (β)

As Theorem 1 indicates, the set of permissible values of $v_{0,n}$ consists of all fractions whose denominators do not exceed n.

For the absolute value of the difference of two unequal permissible rate values $v_1 = p_1/q_1$ and $v_2 = p_2/q_2$ we have the following lower bound:

$$|v_1 - v_2| = |p_1 q_2 - p_2 q_1| / q_1 q_2 \geqslant 1/n^2. \tag{20}$$

Therefore an interval of length $1/n^2$ cannot contain more than one permissible value of $v_{0,n}$.

Consider a jump f of type (0, n) at zero. Assume that we have applied operator P to it t times. Then

$$x_l(P^t f)/t \leqslant v_{0,n} \leqslant x_r(P^t f)/t. \tag{21}$$

Lemma 5 yields that $(x_r(P^t f)/t) - (x_l(P^t f/t) \le nD/t$.

Thus formula (21) shows that $v_{0,n}$ lies in an interval of length nD / t; but a maximum of one permissible $v_{0,n}$ value can be contained in an interval of length $1/n^2$. Consequently, it suffices to take t to be such that we have $nD/t \le 1/n^2$, e.g., to set $t = n^3D$, and then on the interval $(x_l(P^{n^3D}f)/n^3D,\ x_r(P^{n^3D}f)/n^3D)$ to choose a fraction p / q whose denominator does not exceed n.

In accordance with the above, this fraction is unique and equal to $v_{0,n}$. This completes the proof.

4. Estimation of the Number of Operations

Let us determine the number of operations needed to determine the rate $v_{0,n}$ by the method described in Sec. 2. An operation will be understood to be a single-shot computation of the function φ that determines the operator P.

Since Lemma 5 yields that for all t we have $d(P^t f) \le nD$, a single-shot application of operator P to ladder f corresponds to not more than nD operations needed to determine the resultant ladder Pf.

To determine $v_{0,n}$, operator P was applied $t = n^3D$ times to a jump of type (0, n) at zero. Consequently, the number of operations needed to compute $v_{0,n}$ is bounded from above by the number

$$\asymp nD \cdot t = n^4 D^2. \tag{22}$$

(We have ignored actions needed to perform arithmetic operations.)

The proof of Theorem 2 is provided by Paras. 1-4 taken jointly.

APPENDIX

We will obtain some results of [1] by using the techniques developed in Sec. 1. No constraints will be placed on pair (0, n); it can even be uncoupled.

In metric space (R_+^{n-1}, ρ) let us consider cube $\mathscr{K}$ with edge 2D, one of whose vertices is the point 0, while the edges that originate from 0 are located on the positive semiaxes of the coordinate system.

In addition to operator $\mathscr{P}.$ that acts in this space let us define operator $Q.$ (which acts in the same fashion) as follows: If $\Delta = (\Delta_1, \ldots, \Delta_{n-1})$, then we set

$$Q.(\Delta) = (\min(\Delta_1, 2D), \ldots, \min(\Delta_{n-1}, 2D)). \tag{23}$$

Operator $Q.$ is continuous, and therefore $Q.\mathscr{P}.$ is also a continuous operator and carries $\mathscr{K}$ into itself. On the basis of Brauer's theorem, there exists in $\mathscr{K}$ a stationary point ω for $Q.\mathscr{P}.$:

$$Q.\mathscr{P}.\omega = \omega. \tag{24}$$

Assume that vector ω corresponds to ladder g. We apply to ladder $\mathscr{P}g$ the operator Q induced by $Q.$. Here Q leaves all file lengths of $\mathscr{P}g$ that are less than 2D unaltered, and makes all the remaining ones equal to 2D. We have from (24), however, that we obtain the original ladder g as a result. Therefore we have the following.

Proposition 1. Ladder g corresponding to vector ω defined by Eq. (24) is transformed as follows as a result of operator $\mathscr{P}$: For all t the lengths of certain files of $\mathscr{P}^t g$ remain unaltered, while the lengths of all remaining files increase linearly with respect to t.

We can readily obtain (and even somewhat strengthen) the assertions of items β) and δ) of Lemma 5 in [1] on the basis of Proposition 1. Let us formulate them again.

Proposition 2. β) Assume that f is a jump of type (0, n) at zero. Then for all t, to within $O(1)$, the left and right coordinates of ladder $\mathscr{P}^t f$ are given by the following formulas:

$$x_l(\mathscr{P}^t f) = L_{0,n} t + O(1), \tag{25}$$

$$x_r(\mathscr{P}^t f) = R_{0,n} t + O(1), \tag{26}$$

where $|O(1)| \leq nD$.

δ) There exists a unique sequence $0 < c_1 < c_2 < \ldots < c_m < n$, $m \geq 0$, such that pairs $(0, c_1)$, $(c_1, c_2), \ldots,$ (c_m, n) are coupled and $L_{0,n} = v_{0,c_1} \leq v_{c_1,c_2} \leq \ldots \leq v_{c_m,n} = R_{0,n}$, the numbers $c_1, c_2, \ldots, c_m$ being uniquely defined by operator $\mathscr{P}$ and corresponding to those levels of ladder g whose files increase linearly as a result of the action of $\mathscr{P}^t$.

The proof of β) can be obtained by repeated use of assertion 1 and Lemma 5; to prove δ), we must majorize and minorize $\mathscr{P}^t f$ by means of shifts of ladder g.

One final remark: All the results of [1] and of this paper remain valid and can be transferred almost verbatim to the case of a continuous argument (something that was partially utilized in this paper). In particular, Theorem 2 of [1] is valid, and we can take the number nD to be the constant $\mathscr{K}_3$ in its formulation.

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