ONE-DIMENSIONAL AUTOMATON NETWORKS WITH MONOTONIC LOCAL INTERACTION

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The behavior of strings that are infinite in both directions and consist of identical finite automata is investigated. The states of a finite number of adjacent automata provide the input for each automaton. Monotonic interaction of automata is considered. An effective means of predicting the following two aspects of the behavior of such systems is given: a) are they "wash-out" systems and b) how do the states of systems of automata with long initial files vary as $t \to \infty$.

1. INTRODUCTION; BASIC DEFINITIONS AND FORMULATIONS

We will investigate the behavior of one-dimensional networks of identical automata with $n + 1$ states. The automata are located at points of an integer-valued straight line $\mathbf{Z}$, and their states change with time. Time is assumed to be discrete, while the state of any automaton at an arbitrary instant depends only on the states of a finite number of automata at a preceding instant. The way in which such networks function is described by a deterministic monotonic operator P. Criteria are derived that make it possible to predict one property of such operators, and the behavior of some states as $t \to \infty$ is described.

Let us give some exact definitions and formulations. Any mapping $f: \mathbf{Z} \to \{0, 1, 2, \ldots, n\}$ will be called a state. The set of such states will be denoted by $F = \{f\}$.

Definition 1. Assume we are given a natural number r and function $\varphi: \{0, 1, 2, \ldots, n\}^{2r+1} \to \{0, 1, \ldots, n\}$, it being assumed that $\varphi(0, \ldots, 0) = 0$. We define the operator $P: F \to F$ by the equation $(Pf)(x) = \varphi(f(x - r), \ldots, f(x + r))$, $\forall x \in \mathbf{Z}$.

Definition 2. Operator P will be called monotonic if φ is a monotonic function, i.e.,

$$a_{-r} \leqslant a_{-r}', \ldots, a_r \leqslant a_r' \Rightarrow \varphi(a_{-r}, \ldots, a_r) \leqslant \varphi(a_{-r}', \ldots, a_r').$$

Henceforth we will consider only monotonic operators.

The number $D = 2r$ will be called the diameter of operator P.

Definition 3. The level of point $x \in \mathbf{Z}$ for state f is what we will call the number $f(x) \in \{0, 1, \ldots, n\}$.

Definition 4. State f is called an island if there exists only a finite number of points that are on a nonzero level.

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Definition 5. A file of level a is what we will call a segment $K \subset \mathbf{Z}$ (finite or infinite) such that all points $x \in K$ are on one level a, while points adjacent to K are at other levels. The number of points $|K|$ in segment K will be called the length of the file.

Definition 6. We will say that island f is washed out by operator P if there exists a natural number t_0 such that for all natural $t > t_0$, $P^t f = (\ldots 000 \ldots)$ (null distribution).

Definition 7. Operator P is called a wash-out operator if any island can be washed out by P.

In conformity with Definitions 5 and 6. we can say that island f is not washed out by operator P if for all $t = 1, 2, 3, \ldots$, $P^t f \neq (\ldots 000 \ldots)$; we call P a non-wash-out operator if an island that cannot be washed out exists for it.

In the main body of the text, we make the following inessential but convenient assumption: for all $0 \leq a \leq n$

$$P(\ldots aaa \ldots) = (\ldots aaa \ldots).$$

This assumption applies to all the text that follows up to Sec. 2 inclusive. In Sec. 3 we show how to avoid using this assumption.

Definition 8. State f is called monotonically increasing if for all pairs (x_1, x_2), $x_1 < x_2$, $x_i \in \mathbf{Z}$ $(i = 1, 2)$ we have the inequality

$$f(x_1) \leqslant f(x_2).$$

Monotonically decreasing states are similarly defined. All such states are called monotonic. Every monotonic state f that is not a constant will be called a ladder. The quantities $a = \min_{x \in \mathbf{Z}} f(x)$, $b = \max_{x \in \mathbf{Z}} f(x)$ will be called the lowest and highest levels of ladder f (with intermediate levels between them). Ladder f will be called a ladder of type (a, b).

Remark. In view of our assumption, it is easy to see that if f is a ladder of type (a, b), then Pf is also a ladder of type (a, b) (this follows from the fact that P is a uniform and monotonic operator).

Definition 9. Assume we are given ladder f of type (a, b). We denote the coordinate of the rightmost point of line $\mathbf{Z}$ having level a by $x_l(f)$, and the coordinate of the leftmost point of $\mathbf{Z}$ having level b, by $x_r(f)$. The numbers $x_l(f)$, $x_r(f)$ will be called the left and right coordinates of the ladder, respectively.

Remark. If f_1 and f_2 are two ladders of the same type (a, b) and $f_1(x) \geq f_2(x)$ for all $x \in \mathbf{Z}$, then $x_l(f_1) \leq x_l(f_2)$ and $x_r(f_1) \leq x_r(f_2)$.

Definition 10. The length $d(f)$ of ladder f is what we will call $d(f) = x_r(f) - x_l(f)$. We note that we always have $d(f) \geq 1$.

Definition 11. A jump of type (a, b) is what we will call a ladder f of type (a, b) for which $d(f) = 1$ [i.e., $f = (\ldots aabb \ldots)$].

If in this case we have $x_l(f) = c$, we will speak of a jump of type (a, b) with coordinate at point c.

Now we can formulate the following lemma.

LEMMA 1. If f is an arbitrary jump of type (a, b), then the following limits exist:

$$\lim_{t\to\infty} x_l(P^t f)/t = L_{a,b}, \qquad \lim_{t\to\infty} x_r(P^t f)/t = R_{a,b}.$$

Definition 12. We will call $L_{a,b}$ the left (a, b)-rate and $R_{a,b}$, the right (a, b)-rate; clearly, we always have $R_{a,b} \geq L_{a,b}$. If $L_{a,b} = R_{a,b}$, then we denote them by $v_{a,b}$ and call them the (a, b)-rate.

The proof of Lemma 1 will be given in Sec. 2.

Paper [1] provides an economical algorithm for computing $L_{a,b}$ and $R_{a,b}$.

Theorems 1 and 2 that follow comprise the basic result of this paper. The first of them provides a criterion for determining whether P is a wash-out operator, while the second describes the behavior of a fairly large class of ladders as $t \to \infty$.

THEOREM 1. Operator P is a non-wash-out operator if and only if at least one of the following inequalities holds:

$$R_{0,n} \leqslant L_{n,0};\ R_{0,n-1} \leqslant L_{n-1,0};\ \ldots;\ R_{0,2} \leqslant L_{2,0};\ R_{0,1} \leqslant L_{1,0}. \tag{1}$$

Before formulating Theorem 2, we will describe the following geometrical procedure.

Procedure. We denote by $\mathfrak{A}$ the set of all ladders of type (a, b), $b - a = M$, for which the levels of any two adjacent files differ from one another by 1. We consider an arbitrary $f \in \mathfrak{A}$.

Further, we consider a Cartesian half-plane with coordinates (x, t), $t \geq 0$. On the x axis we lay out the coordinate $x_1, \ldots, x_M$ of all points that separate the boundaries of files of ladder f and denote the levels of these files by $a_0 = a, a_1, a_2, \ldots, a_M = b$. From points $x_1, \ldots, x_M$ we draw arcs on the half-plane in question with angular coefficients that are equal to $1/v_{a_i, a_{i+1}}$, respectively, for point x_i $(i = 1, \ldots, M)$.

We denote by t_1 the minimum t value such that any two of these rays (departing from points x_i and x_{i+1}) intersect. We eliminate the continuations of these rays past the point of intersection and instead we draw rays from this point with angular coefficient $1/v_{a_i, a_{i+2}}$ (if $a_i = a_{i+2}$, we do not do so); the rate $v_{a_i, l_{i+2}}$ clearly exists in this case [this being a consequence of Lemma 5 (γ)]. If several pairs of rays intersect for $t = t_1$ (or more than two rays intersect), we proceed analogously.

Taking t_1 as a new point for reckoning time, we repeat the process, and so on. Obviously, by employing this process not more than M times, for some $t = T$ we will arrive at a situation in which no two rays intersect. We terminate the process at this point.

To each resultant portion of the half-plane we assign the level number of the file that is on straight line $t = 0$ at the point at which this portion of the half-plane is in contact with it (the file in question is nonempty for each portion).

This completes the procedure.

THEOREM 2. For every operator P there exist constants $\mathcal{K}_1$ and $\mathcal{K}_2$ that depend only on P and the numbers a and b and are such that if for a fixed ladder $f \in \mathfrak{A}$ of type (a, b) all file lengths exceed $\mathcal{K}_1$, then to within an additive constant $\mathcal{K}_2$ points that delimit the boundaries of files for ladder $P^t f$ will have (for all t) coordinates that are defined in terms of the coordinates of the corresponding points of ladder f by the geometrical procedure described above.

More precisely, we can assert the following regarding the components of $P^{t'} f$.

a) Assume that segment $[z_1, z_2]$ is the intersection of straight line $t = t'$ and the portion of the half-plane to which level l is ascribed. Then all state components of $P^{t'} f$ with coordinates x, where $z_1 + \mathcal{K}_2 \leq x \leq z_2 - \mathcal{K}_2$, have level l.

b) All remaining components of $P^{t'} f$ have levels that are not less than the smaller and not greater than the greater of the component levels that are closest to them with respect to coordinate $x \in \mathbf{Z}$ on the left and the right, and defined by condition a).

c) If $t' > T$, then the set of levels of those files of $P^{t'} f$, which increase linearly with respect to time for $t > t'$, is uniquely defined by the geometrical procedure described above and is independent of the initial relationship of the file lengths.

d) This set can be described as the minimum sequence (in lengths) of the form $a < c_1 < \ldots < c_m < b$ (where a and b are the lowest and highest levels of the ladder) for which all pairs (a, c_1), (c_1, c_2), ..., (c_{m-1}, c_m) are coupled (see Definition 13). Here the rates $v_{a, c_1}, v_{c_1, c_2}, \ldots, v_{c_m, b}$ exist and satisfy the inequalities $v_{a, c_1} \leq v_{c_1, c_2} \leq \ldots \leq v_{c_m, b}$.

e) For $t' > T$ the length of a file of level c_s $[1 \leq s \leq m$; see item d)] is given by the formula $\Delta_{i_s} = (v_{c_s, c_{s+1}} - v_{c_{s-1}, c_s})t' + O(1)$, while the length of ladder $P^{t'} f$ is given by

$$d(P^{t'} f) = \sum_{s=1}^{m} \Delta_{i_s} + O(1) = (R_{a,b} - L_{a,b})t' + O(1),$$

where $|O(1)| \leq \mathcal{K}_3$, with $\mathcal{K}_3$ being a constant that depends only on the operator P, the numbers a and b, and the original ladder f.

2. PROOFS

Figures 1 and 2 show separately the overall plan of the proofs and the plan for the proof of Lemma 5 (by induction). The arrows show which assertions are employed in proving the corresponding assertion.

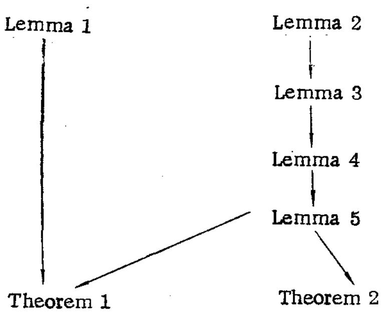

Fig. 1. Overall plan of proofs.

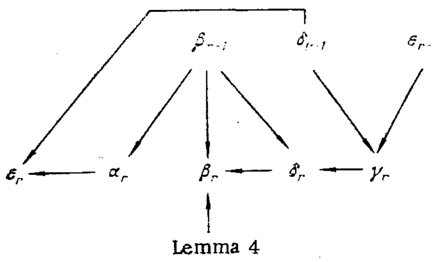

Fig. 2. Plan of proof of Lemma 5.

Let us first prove Lemma 1. In the course of the proof, we will employ the following lemma.

SEQUENCE LEMMA. Assume that A_t is some numerical sequence, where for all t and τ

$$A_{t+\tau} \leqslant A_t + A_\tau. \tag{2}$$

Then for any t_0 we have $A_{t_0}/t_0 \geqslant \overline{\lim_{t\to\infty}} A_t/t$. If, however, for all t and τ we have the inequality

$$A_{t+\tau} \geqslant A_t + A_\tau,$$

then for any t_0 we have $A_{t_0}/t_0 \leqslant \underline{\lim_{t\to\infty}} A_t/t$.

COROLLARY OF SEQUENCE LEMMA. When any of the above inequalities holds, the limit $\lim_{t\to\infty} A_t/t$ exists.

Proof of Sequence Lemma. Assume that inequality (2) holds. We fix an arbitrary natural t_0 and consider any natural N. Let $N = mt_0 + t_1$, $0 \le t_1 < t_0$, and let m be some natural number. Then

$$A_N/N = N^{-1}A_{mt_0+t_1} \leqslant N^{-1}(mA_{t_0} + A_{t_1}) = N^{-1}(A_{t_1} + (m + t_1t_0^{-1})A_{t_0}) - N^{-1}(t_1t_0^{-1})A_{t_0} = N^{-1}(A_{t_1} - A_{t_0}t_1t_0^{-1}) + A_{t_0}t_0^{-1}.$$

The expression in parentheses in the first term is bounded by a constant. Letting $N \to \infty$, we obtain what was required. The proof is entirely analogous when the second inequality holds.

Proof of Lemma 1. P^tf is a ladder with lower and upper levels a and b and left and right coordinates $x_l(P^tf)$, $x_r(P^tf)$. We enclose the ladder between two jumps of type (a, b) with coordinates $x_l(P^tf)$ and $x_r(P^tf)$:

$$S^{x_l(P^tf)}f \geqslant P^tf \geqslant S^{x_r(P^tf)}f \tag{3}$$

[we recall that $S^kf(x) = f(x - k)$].

The fact that P is a monotonic operator implies that if we apply operator P^τ to all parts of inequality (3), we obtain $P^\tau S^{x_l(P^tf)}f \ge P^{t+\tau}f \ge P^\tau S^{x_r(P^tf)}f$. But P is a uniform operator, and therefore we have

$$S^{x_l(P^tf)}P^\tau f \geqslant P^{t+\tau}f \geqslant S^{x_r(P^tf)}P^\tau f. \tag{4}$$

Setting $t = \tau$ in inequality (3), we will have

$$S^{x_l(P^\tau f)}f \geqslant P^\tau f \geqslant S^{x_r(P^\tau f)}f. \tag{5}$$

We now replace $P^\tau f$ in equality (4) by its majorant and minorant from (5):

$$S^{x_l(P^tf)}(S^{x_l(P^\tau f)}f) \geqslant P^{t+\tau}f \geqslant S^{x_r(P^tf)}(S^{x_r(P^\tau f)}f)$$

or

$$S^{x_l(P^tf)+x_l(P^\tau f)}f \geqslant P^{t+\tau}f \geqslant S^{x_r(P^tf)+x_r(P^\tau f)}f. \tag{6}$$

The remark associated with Definition 9, and inequality (6), imply that the following inequalities hold:

$$x_l(P^tf) + x_l(P^\tau f) \leqslant x_l(P^{t+\tau}f), \tag{7}$$

$$x_r(P^{t+\tau}f) \leqslant x_r(P^tf) + x_r(P^\tau f). \tag{8}$$

We apply the corollary to the sequence lemma to the sequences $x_l(P^tf)$ and $x_r(P^tf)$.

It follows from inequality (7) that the limit $\lim_{t\to\infty} x_l(P^tf)/t = L_{a,b}$ exists, while inequality (8) yields that $\lim_{t\to\infty} x_r(P^tf)/t = R_{a,b}$ exists.

This completes the proof of Lemma 1.

Proof of Theorem 1. (In the forward direction.) We will prove that if for all $l \geq 1$ we have $R_{0,l} > L_{l,0}$, then P is a wash-out operator. By induction on l we can prove that for every $l = 0, 1, \ldots, n$ the island f = $(\ldots 000lll \ldots ll000 \ldots)$ is washed out.

This is obvious for $l = 0$. Assume that for all $l \leq l_0 - 1$ the island $(\ldots 00l \ldots l00 \ldots)$ is washed out. Consider island g = $(\ldots 000 l_0 l_0 \ldots l_0 000 \ldots)$. If for it the length of the file of level l_0 is d, then over a time $t = d/(R_{0,l_0} - L_{l_0,0})$ the level of each point of island $P^t g$ will be less than l_0, and therefore $P^t g$ can be majorized by an island of the form $(\ldots 00(l_0 - 1)(l_0 - 1) \ldots (l_0 - 1)00 \ldots)$, which is washed out by the inductive assumption. The inductive process is thus complete; island g is washed out.

Now if h is an arbitrary island for which $\max_{x \in Z} h(x) \leqslant l$, then h can be majorized by an island of the form $(\ldots 00ll \ldots ll00 \ldots)$, and will therefore we washed out by P (the monotonic property of P has been employed).

The notion of coupling of a pair of levels will play an important part in what follows.

Definition 13. Pair (a, b) is called coupled if there exist natural numbers L and T, such that for any ladder f of type (a, b)

$$d(f) \geqslant L \Rightarrow d(P^T f) \leqslant d(f) - 1.$$

Numbers L and T will be called (a, b)-coupling parameters.

Remark. Obviously, pair $(a, a+1)$ is coupled. For it we can take $L = 2$, $T = 1$.

LEMMA 2. Assume that (a, b) is a coupled pair with parameters L and T, and assume that f is an arbitrary ladder of type (a, b). Then there exists a t_0 such that for all $t \geq t_0$ we have $d(P^t f) \leq L + DT$ (we recall that $D = 2r$ is the diameter of a neighborhood of point $0 \in \mathbf{Z}$).

Remark. The number $L + DT$ will be called the coupling constant for pair (a, b). If $d(f) < L$, then we can set $t = 0$.

It suffices to carry out the proof for the case $d(f) < L$. Indeed, we will show that there necessarily exists an instant at which the length of the ladder will be strictly less than L. If $d(f) \geq L$, then in view of the fact that pair (a, b) is coupled we have $d(P^{lT} f) \leq d(P^{(l-1)T} f) \leq \ldots \leq d(f) - l$; taking $l > d(f) - L$. we obtain $d(P^{lT} f) < L$.

Thus, assume that for $t = 0$ we have $d(f) < L$. Consider $t_1 = \min\{t : d(P^t f) > L + DT\}$. Clearly, $t_1 > T$, since the ladder length cannot increase by more than D per unit time, and $d(f)$ cannot increase by more than TD over a time $t < T$. Consider the instant $t_1 - T$. It follows from the definition of t_1 that $d(P^{t_1 - T} f) \leq L + DT$.

If $d(P^{t_1 - T} f) < L$, then $d(P^{t_1} f) < L + DT$, since the ladder length could not increase by more than DT over time T. This contradicts our definition of t_1.

Therefore, $d(P^{t_1 - T} f) \geq L$. In view of the fact that pair (a, b) is coupled, $d(P^{(t_1 - T) + T} f) \leq d(P^{t_1 - T} f) - 1 \leq (L + DT) - 1 < L + DT$.

Lemma 2 has thus been proved.

LEMMA 3. If (a, b) is a coupled pair, then $L_{a,b} = R_{a,b}$, and hence rate $v_{a,b}$ exists.

Proof. If f is a jump of type (a, b), then $d(f) = 1$ and from Lemma 2 we have that, for all $t \geq 0$, $x_r(P^t f) - x_l(P^t f) < L + DT$. Therefore, $(x_r(P^t f)/t) - (x_l(P^t f)/t) \to 0$, $t \to \infty$.

From this and from Lemma 1 we have $L_{a,b} = R_{a,b} (= v_{a,b})$.

Remark. The existence of (a, b)-rate $v_{a,b}$ still does not yield that (a, b) is a coupled pair. For example, for the identity operator $P : Pf \equiv f$ all rates $v_{a,b}$ exist and at the same time all pairs (a, b), where $|a - b| \geq 2$, are not coupled.

The following important lemma is a corollary of Lemma 3.

LEMMA 4. If (a, b) is a coupled pair and f is a jump of type (a, b) at zero, then for any natural t we have $(P^t f)/t \leq v_{a,b} \leq x_r(P^t f)/t$, and therefore

$$|x_l(P^t f) - v_{a,b} t| \leqslant L + DT,$$
$$|x_r(P^t f) - v_{a,b} t| \leqslant L + DT.$$

The proof of Lemma 4 is obvious.

The following lemma (Lemma 5) plays a major part in the proof of Theorem 1 and Theorem 2, and occupies the main part of Sec. 2.

LEMMA 5. Assume that $a < b$ are any two levels.

We have the following five assertions.

(α) Pair (a, b) is coupled if and only if for any c, $a < c < b$, we have the inequality $L_{a,c} > R_{c,b}$.

(β) There exists a constant $k(a, b)$ such that if f is a jump of type (a, b) with coordinates at 0, then for all natural N we have

$$|x_l(P^N f) - L_{a,b}\cdot N| < k(a, b),$$
$$|x_r(P^N f) - R_{a,b}\cdot N| < k(a, b)$$

(the left and right coordinates of ladder $P^N f$ move "almost linearly").

(γ) If $a < c < b$, pairs (a, c) and (c, b) are coupled, and $v_{a,c} > v_{c,b}$, then (a, b) is a coupled pair, and therefore rate $v_{a,b}$ exists, where $v_{c,b} \le v_{a,b} \le v_{a,c}$.

(δ) The sequence $a < c_1 < \ldots < c_m < b$, $m \ge 0$, such that pairs (a, c_1), (c_1, c_2), . . . , (c_m, b) are coupled and $L_{a,b} = v_{a,c_1} \le v_{c_1,c_2} \le \ldots \le v_{c_m,b} = R_{a,b}$, exists and is unique, the numbers $c_1, \ldots, c_m$ being uniquely defined by operator P and numbers a and b.

(ε) Pair (a, b) is coupled if and only if for any c_1 and c_2 such that $a < c_1 < c_2 < b$ and pairs (a, c_1), (c_2, b) are coupled, we have the inequality $v_{a,c_1} > v_{c_2,b}$.

Similar assertions hold for $a > b$.

Proof of Lemma 5. We will carry out the proof by induction on the difference $r = b - a$. For a given r we furnish assertions (α)-(ε) with a subscript r; they thus become assertions (α_r)-(ε_r), which we will prove by induction on r.

1. Basis of the Induction. If $r = 1$, i.e., $b = a + 1$, then assertions (α_1), (γ_1), and (ε_1) hold automatically since there are no integers in the interval between a and $a + 1$. Assertion (β_1) holds, since $x_l(P^N f) = v_{a,a+1}N$, $x_r(P^N f) = v_{a,a+1}N + 1$. Assertion ($\delta_1$) is valid; here $m = 0$, while $(a, a + 1)$ is a coupled pair.

2. Induction Step. Assume that Lemma 5 has been proved for $b - a < r$. We will prove it for $b - a = r$.

1. First we will prove that assertion (α_r) implies (ε_r).

For this it suffices to prove the following assertion: if for any c, $a < c < b$, we have the inequality $L_{a,c} > R_{c,b}$, then for any c_1 and c_2 such that $a < c_1 < c_2 < b$, the pairs (a, c_1), (c_2, b) being coupled, we have the inequality $v_{a,c_1} > v_{c_2,b}$.

In proving this assertion we will utilize assertion (δ_{r-1}), which holds by virtue of the inductive assumption, and the remark made in Definition 9.

Assume that for any c, $a < c < b$, we have $L_{a,c} > R_{c,b}$. Then for any c_1 and c_2 such that $a < c_1 < c_2 < b$ and pairs (a, c_1), (c_2, b) are coupled, we have the following inequality by virtue of the remark in Definition 9: $v_{a,c_1} = L_{a,c_1} > R_{c_1,b} \ge R_{c_2,b} = v_{c_2,b}$; the assertion is thus proved.

2. Let us prove assertion (α_r) in one direction; specifically, if (a, b) is a coupled pair, then for any c, $a < c < b$, we have the inequality $L_{a,c} > R_{c,b}$. In the proof we will use assertion (β_{r-1}) and the remark in Definition 9.

Assume the contrary; assume that there exists a c, $a < c < b$, such that $L_{a,c} \le R_{c,b}$. Assertion (β_{r-1}) implies that there exist constants $k(a, c)$ and $k(c, b)$ such that

$$|x_l(P^N f_1) - L_{a,c}\cdot N| < k(a, c),$$
$$|x_r(P^N f_1) - R_{a,c}\cdot N| < k(a, c), \quad (9)$$

$$|x_l(P^N f_2) - L_{c,b}\cdot N| < k(c, b),$$
$$|x_r(P^N f_2) - R_{c,b}\cdot N| < k(c, b), \quad (10)$$

where f_1 is a jump of type (a, c) with coordinate at 0, while f_2 is a jump of type (c, b) with coordinate at 0.

We fix a number $\mathscr{K} > 0$ and consider the "ladder" f(x):

$$f = \begin{cases} a & \text{for} \quad x<0, \\ c & \text{for} \quad 0 \leqslant x \leqslant \mathscr{K}, \\ b & \text{for} \quad x>\mathscr{K}, \end{cases}$$

and also the jumps $g_1(x)$ and $g_2(x)$:

$$g_1 = \begin{cases} a & \text{for} \quad x<0, \\ c & \text{for} \quad x \geqslant 0, \end{cases}$$

$$g_2 = \begin{cases} c & \text{for} \quad x \leqslant \mathscr{K}, \\ b & \text{for} \quad x>\mathscr{K}, \end{cases}$$

where $g_1(x) < f(x) < g_2(x)$ for all $x \in \mathbf{Z}$. The remark in Definition 9 and inequalities (9) and (10) imply that for any N we have the following inequalities:

$$x_l(P^N f) \leqslant x_l(P^N g_1) \leqslant L_{a,c} \cdot N + k(a, c), \tag{11}$$

$$x_r(P^N f) \geqslant x_r(P^N g_2) \geqslant (R_{c,b} \cdot N + \mathscr{K}) - k(c, b). \tag{12}$$

From this, in view of our assumption that $L_{a,c} \leq R_{c,b}$, we have

$$d(P^N f) = x_r(P^N f) - x_l(P^N f) \geqslant \mathscr{K} - k(a,c) - k(s,b) + (R_{c,b} - L_{a,c})N \geqslant \mathscr{K} - k(a,c) - k(c,b). \tag{13}$$

Because of the arbitrary way in which $\mathscr{K}$ is chosen, we obtain from (13) that the length $d(P^N f)$ cannot be bounded by a constant L + DT, and then Lemma 2 yields that (a, b) is not a coupled pair. We have thus obtained a contradiction with the coupled nature of (a, b). Therefore, for all c, $a < c < b$, we have $R_{c,b} < L_{a,c}$, and we have proved assertion (α_r) in the forward direction.

3. We will prove assertion (α_r) in the reverse direction; specifically, if for any c, $a < c < b$ we have $L_{a,c} > R_{c,b}$, then (a, b) is a coupled pair. In the proof we will utilize assertion (β_{r-1}). It follows from (β_{r-1}) that there exist constants $k(a, c)$ and $k(c, b)$ such that inequalities (9) and (10) hold. Let $k = \max\limits_{a<c<b}(k(a, c), k(c, b))$ and $\delta = \min\limits_{a<c<b} |R_{c,b} - L_{a,c}|$. Consider a T such that $T\delta - 2k = \Delta > 1$. We will show that then we can take the coupling parameters for pair (a, b) to be numbers T (the exponent of operator P in Definition 13) and $L = 2TD(b - a)$.

Assume that f is a ladder of type (a, b) and $d(f) \geq L$. Then there exists a c, $a < c < b$, such that the levels of points of some segment of length not less than $2TD = L/(b - a)$ are the same and equal to c. [Indeed, if for each c the length of a file of level c were to be less than 2TD, we would have $d(f) < L$, something that is not the case.]

Consider auxiliary jumps $g_1(x)$ and $g_2(x)$, $x \in \mathbf{Z}$:

$$g_1 = \begin{cases} a & \text{for} \quad x \leqslant x_l(f), \\ c & \text{for} \quad x > x_r(f). \end{cases} \qquad g_2 = \begin{cases} c & \text{for} \quad x < x_l(f), \\ b & \text{for} \quad x \geqslant x_r(f). \end{cases}$$

We have $g_1(x) \geq f(x)$ for $x \leq x_1$ (x_1 is the right coordinate and x_0 the left coordinate of a file of level c). From this we have

$$P^T g_1(x) \geqslant P^T f(x), \quad x \leqslant x_1 - TD. \tag{14}$$

Since, after T-fold application of operator P, the left coordinate of a file of level c could not shift by more than TD, while

$$x_1 - x_0 \geqslant 2TD, \tag{15}$$

we have

$$x_l(P^T g_1) \leqslant x_0 + TD \leqslant x_1 - TD; \tag{16}$$

and hence formula (14) is applicable.

Formula (9) and the remark in Definition 9 yield the following inequality:

$$x_l(P^T f) \geqslant x_l(P^T g_1) \geqslant x_l(f) + L_{a,c} T - k(a,c). \tag{17}$$

Similar reasoning permits us to write

$$x_r(P^T f) \leqslant x_r(P^T g_2) \leqslant x_r(f) + R_{c,b} T + k(c,b). \tag{18}$$

Consequently,

$$d(P^{\tau}f)=x_{\mathrm{r}}(P^{\tau}f)-x_{l}(P^{\tau}f)\leqslant(x_{\mathrm{r}}(f)-x_{l}(f))-T(L_{a,c}-R_{c,b})+k(c,b)+k(a,c)\leqslant d(f)-\Delta<d(f)-1. \tag{19}$$

We obtain from (19) that (a, b) is a coupled pair with coupling parameters T and L. Assertion (α_r) is thus fully proved.

4. Let us prove assertion (γ_r). We will employ assertions (δ_{r-1}) and (ε_{r-1}) in the proof.

Assume that (a, c) and (c, b) are coupled pairs and $v_{a,c} > v_{c,b}$. We will show that for any c_1, c_2, $a < c_1 < c_2 < b$ such that (a, c_1) and (c_2, b) are coupled pairs, we have $v_{a,c_1} > v_{c_2,b}$. Then we immediately obtain from (ε_{r-1}) that (a, b) is a coupled pair, and thus we have proved assertion (γ_r).

However, let us assume the contrary: assume that for some c_1 and c_2, $a < c_1 < c_2 < b$, (a, c_1), (c_2, b) are coupled pairs, but

$$v_{a,c_1}\leqslant v_{c_2,b}. \tag{20}$$

The following cases are possible.

<u>Case 1.</u> $c_1 \le c \le c_2$. Then

$$\begin{aligned} v_{a,c_1}&=L_{a,c_1}\geqslant v_{a,c},\\ v_{c,b}&=R_{c,b}\geqslant R_{c_2,b}=v_{c_2,b} \end{aligned} \tag{21}$$

(this follows from the remark in Definition 9). By the condition of item (γ_r), $L_{a,c} = v_{a,c} > v_{c,b} = R_{c,b}$. We obtain from (21) that $v_{a,c_1} > v_{c_2,b}$, something that contradicts our assumption (20).

<u>Case 2.</u> $c_1 \le c_2 < c$.

<u>Case 3.</u> $c < c_1 \le c_2$.

Cases 2 and 3 are analogous to one another. For instance, let us consider Case 2.

It follows from assertion (δ_{r-1}) that the segment $[c_2, c]$ can be partitioned by points $c_2 < q_1 < \dots < q_s < c$ such that

$$v_{c_2,q_1}\leqslant v_{q_1,q_2}\leqslant\ldots\leqslant v_{q_s,c}. \tag{22}$$

Since the remark in Definition 9 yields that $v_{c_2,b} \le v_{c_2,q_1}$, if we take inequality (20) into account, we can extend chain (22):

$$v_{a,c_1}\leqslant v_{c_2,b}\leqslant v_{c_2,q_1}\leqslant v_{q_1,q_2}\leqslant\ldots\leqslant v_{q_s,c}. \tag{23}$$

From (23) we obtain

$$v_{a,c_1}\leqslant v_{q_s,c}. \tag{24}$$

But assertion (ε_{r-1}) yields that since (a, c) is a coupled pair and $a < c_1 < q_s < b$, we have $v_{a,c_1} > v_{q_s,b}$, and this contradicts inequality (24) (if subscript s for q_s is 0, then c_2 functions as q_s).

Thus our assumption (20) yields a contradiction with the coupled nature of the situation, in the form of inequality (24). Consequently, inequality (20) is contradictory, and assertion (γ_r) has thus been proved.

5. We will prove assertions (δ_r) and (β_r). In proving (δ_r) we will employ assertions (γ_r), (β_{r-1}), and (δ_{r-1}) that have already been proved. In proving (β_r) we will employ the now-proved assertion (δ_r), assertion (β_{r-1}), and the assertion of Lemma 4.

<u>Proof of Assertion (δ_r).</u> We take a sequence of coupled pairs (a, c_1), (c_1, c_2), ..., (c_m, b) of minimum length m $(a < c_1 < c_2 < \dots < c_m < b)$. We write $a = c_0$, $b = c_{m+1}$.

If $v_{c_i,c_{i+1}} \gg v_{c_{i+1},c_{i+2}}$, then by (γ_{r-1}) pair (c_i, c_{i+2}) is a coupled pair, something that contradicts the choice of minimum number m. Consequently

$$v_{a,c_1}\leqslant v_{c_1,c_2}\leqslant\ldots\leqslant v_{c_m,b}; \tag{25}$$

and part of assertion (δ_r) has been proved.

We will show that

$$L_{a,b}=v_{a,c_1},\quad R_{a,b}=v_{c_m,b}. \tag{26}$$

Assertion (β_{r-1}) yields inequalities (9) and (10), in which we have set $c = c_1$; thus, $k(a, c_1)$, $k(c_1, b)$ have been defined. Assume that the number $\mathscr{K}$ is such that

$$\mathscr{K}>k(a, c_1)+k(c_1, b)+D. \tag{27}$$

Consider the following ladder f(x):

$$f=\begin{cases} a & \text{for } x\leqslant 0,\\ c_1 & \text{for } 0<x<\mathscr{K},\\ b & \text{for } x\geqslant\mathscr{K},\end{cases}$$

and jumps $g_1(x)$ and $g_2(x)$:

$$g_1=\begin{cases} a & \text{for } x\leqslant 0,\\ c_1 & \text{for } x>0,\end{cases}$$
$$g_2=\begin{cases} c_1 & \text{for } x<\mathscr{K},\\ b & \text{for } x\geqslant\mathscr{K}.\end{cases}$$

We have $g_1(x) \leq f(x) \leq g_2(x)$ for all $x \in Z$.

We will show that for any T we have the inequality

$$x_r(P^Tg_1)<x_l(P^Tg_2)-D. \tag{28}$$

Indeed, expressions (9) and (10) yield

$$\begin{aligned} x_r(P^Tg_1)&\leqslant R_{a, c_1}T+k(a, c_1),\\ x_l(P^Tg_2)&\geqslant \mathscr{K}+L_{c_1, b}T-k(c_1, b), \end{aligned} \tag{29}$$

from which we have that

$$x_r(P^Tg_1)-x_l(P^Tg_2)\leqslant (R_{a, c_1}-L_{c_1, b})T-(\mathscr{K}-k(a, c_1)-k(c_1, b)). \tag{30}$$

From (27) we obtain $\mathscr{K} - k(a, c_1) - k(c_1, b) > D$ and therefore expression (30) yields that

$$x_r(P^Tg_1)<x_l(P^Tg_2)-D+(R_{a, c_1}-L_{c_1, b})T\leqslant x_l(P^Tg_2)-D, \tag{31}$$

since $R_{a,c_1} - L_{c_1,b} = v_{a,c_1} - v_{c_1,b} \leq 0$. Inequality (28) has thus been proved.

From (28), by induction on T, we obtain that

$$\begin{aligned} &\text{if } x<x_r(P^Tg_1), \text{ then } P^Tf(x)=P^Tg_1(x),\\ &\text{if } x\geqslant x_r(P^Tg_1), \text{ then } P^Tf(x)=P^Tg_2(x). \end{aligned} \tag{32}$$

Therefore, expression (32) yields

$$x_l(P^Tf)=x_l(P^Tg_1)=v_{a, c_1}T+O(1), \text{ where } |O(1)|<\mathscr{K}. \tag{33}$$

Expression (32) also yields

$$x_r(P^Tf)=x_r(P^Tg_2)=R_{c_1, b}T+O(1), \text{ where } |O(1)|<\mathscr{K}. \tag{34}$$

But $R_{c_1,b} = v_{c_m,b}$, by assertion (δ_{r-1}), and therefore from (33) we have $L_{a,b} = v_{a,c_1}$, while (34) yields $R_{a,b} = R_{c_1,b} = v_{c_m,b}$. The equations in (26) have thus been proved.

It remains to show that numbers $c_1, \ldots, c_m$ are uniquely defined by operator P and numbers a and b. If $a \leq c < d \leq b$, where pair (c, d) is coupled, while g is an arbitrary ladder of type (c, d), then those levels of P^Tg which lie above level c and below level d, beginning with some T, have a length that is bounded by a constant (see Lemma 2). This constant depends only on c and d, but not on a, b, or ladder g. Therefore, in investigating the behavior of P^Tf, where f is a ladder of type (a, b) which for each c, $a < c < b$, assumes the value c on a sufficiently large segment, we see that these segments reduce on exactly the differences of the levels (a, c_1), (c_1, c_2), . . . , (c_m, b) $(m \geq 0)$.

Consequently, (a, c_1), . . . , (c_m, b) are coupled pairs and the rates $v_{a,c_1} \leq v_{c_1,c_2} \leq \ldots \leq v_{c_m,b}$ are uniquely defined by operator P and numbers a and b. Assertion (δ_r) has now been fully proved.

The proof of assertion (β_r) follows by combining (33) and (34), in which we respectively set

$$v_{a,c_1}=L_{a,b} \text{ and } v_{c_1,b}=R_{a,b},$$

assertion (β_{r-1}), and Lemma 4 [the latter is significant when (a, b) is a coupled pair]. Thus, (β_r) has also been proved.

Lemma 5 has now been fully proved.

Remark. Assertion (β) will play a significant part in the proof of the main criterion (Theorem 1), although the assertions of the remaining items will be used in proving Theorem 2.

Proof of Theorem 1. (In the reverse direction.) Assume that for some l we have the inequality $R_{0,l} \leq L_{l,0}$. Assume that f is a jump of type (0, l) at zero, while g is a jump of type (l, 0) at zero. From the assertion of item (β) of Lemma 5 we have that there exist constants $k_1 = k(0, l) > 0$, $k_2 = k(l, 0) > 0$ such that for all $t > 0$

$$x_r(P^t f) < R_{0,l} t + k_1, \tag{35}$$

$$x_l(P^t g) > L_{0,l} t - k_2. \tag{36}$$

Consider an island h = (. . . 000ll. . . ll000 . . .) for which the length of the file of level l $|h| > k_1 + k_2$. Then for any t the length $|P^t h|$ of a file of level l for island $P^t h$ will be not less than

$$|P^t h| \geqslant (L_{l,0} - R_{0,l})t + |h| - (k_1+k_2) \geqslant |h| - (k_1+k_2).$$

Consequently, island h will not be washed out and P is a non-wash-out operator. Theorem 1 has thus been fully proved.

The proof of Theorem 2 can readily be carried out by induction on the number of steps in the construction (see the description of the geometrical procedure that precedes Theorem 2), using the assertions of Lemma 5.

3. ADDITIONAL REMARKS

1. In the proofs of the lemmas and theorems it was supposed that all distributions of the form (. . . $aaaa$. . .) ($0 \leq a \leq n$) for operator P are invariant.

We will explain how to avoid this condition [the a priori unique condition that remains in force is that the distributions (. . . 000 . . .) and (. . . nnn . . .) are invariant].

Assume that $\alpha_1 < \alpha_2 < \ldots < \alpha_s$ ($0 < \alpha_i < n$) are such that for all i = 1, 2, ... s the distributions

$$\{(\ldots\alpha_i\alpha_i\alpha_i\alpha_i\ldots)\} \tag{37}$$

are stationary points of operator P on the set of all distributions; then all other distributions of the form (. . . aaa . . .) become (under the action of some power of P) one of the distributions (37), the levels of all intermediate distributions forming a monotonic sequence.

Thus, all numbers 0, 1, 2, . . . , n can be partitioned into several groups (s + 2) (0), (α_1), . . . , (α_s), (n), in each of which there is a unique stationary point (. . . $\alpha_i\alpha_i\alpha_i$ . . .) such that any distribution (. . . bbb. . .) from the corresponding group becomes, after some applications of P, the distribution (. . . $\alpha_i\alpha_i\alpha_i$ . . .).

It is easy to see that if a file of level b ((. . . bbb . . .) $\in$ (α_i)) for a ladder is sufficiently long, after repeated application of P to the ladder the length of the level-b file will become less than 2D and will never again exceed 2D. Consequently, it is only for numbers 0, α_1, α_2, . . . , α_s, n (which we will call "significant" numbers) that we need to employ the concepts of coupling of pairs (α_i, α_j) and also the right and left rates R_{α_i,α_j}, L_{α_i,α_j}.

All the lemmas and Theorems 1 and 2 above remain valid; we need only replace the values of all levels by "significant" numbers.

2. Rates $L_{a,b}$ and $R_{a,b}$ are rational numbers (for all a, b). Indeed, if (a, b) is a coupled pair with coupling constant L + DT, while f is a jump of type (a, b), then, as Lemma 2 implies, for all $t \geq 0$ we have $d(P^t f) \leq L + DT$. According to Dirichlet's principle, there exist integers $t_1 < t_2$ and k for which we have $P^{t_2}f = P^{t_1}S^k f$, from which the rate $v_{a,b} = k/(t_2 - t_1) \in \mathbf{Q}$ (where $\mathbf{Q}$ is a field of rational numbers).

If (a, b) is not a coupled pair, then Lemma 5(δ) yields that $L_{a,b} = v_{a,c}$ and $R_{a,b} = v_{d,b}$, where $a < c < d < b$ and pairs (a, c) and (d, b) are coupled. Therefore, $L_{a,b}$, $R_{a,b} \in Q$ as well. From this we can perceive an algorithm for computing $L_{a,b}$, $R_{a,b}$ (a very uneconomical one, to be sure).

TABLE 1

	0	1	2
0	$0\to0$ $1\to0$ $2\to1$	$0\to0$ $1\to1$ $2\to1$	$0\to0$ $1\to1$ $2\to1$
1	$0\to0$ $1\to0$ $2\to1$	$0\to1$ $1\to1$ $2\to1$	$0\to1$ $1\to1$ $2\to1$
2	$0\to1$ $1\to2$ $2\to2$	$0\to1$ $1\to2$ $2\to2$	$0\to1$ $1\to2$ $2\to2$

Note. The action of operator P is determined from the state of the automaton and its two neighbors (left and right). For fixed states we determine the cell in the table at the intersection of the corresponding row (whose number is the state of the left neighbor) and the column (whose number is the state of the right neighbor), and, depending on the state of the automaton itself, we can uniquely determine the results of the operator (automaton's state at succeeding time instant). It is easy to see that a monotonic operator is specified by the table. It is one of the simplest ones whose examination illustrates the notions introduced in the article (rates, pair coupling, and so forth), as well as the results.

3. As the formulation of the criterion (Theorem 1) indicates, it is natural to introduce a more detailed concept than "washoutability."

Definition 6.3. We will say that island f is unstable under the action of operator P, if there exists a t_0 such that for all $t > t_0$

$$\max_{x\in Z} (P^t f)(x) < \max_{x\in Z} f(x).$$

Then the criterion will be as follows:

Criterion. For operator P there exists a stable island f with $\max_{x\in Z} f(x)=a$ if and only if we have the inequality $R_{0,a} \leq L_{a,0}$.

4. Assume that $f \in \mathfrak{L}$ is an island for which the file levels first increase monotonically from 0 to k, then decrease monotonically from k to 0, while the file lengths exceed $\mathcal{K}_1$. If $R_{0,k} \leq L_{k,0}$ (P a non-wash-out operator), then the overall file length of island $P^t f$ with nonzero levels is (as $t \to \infty$)

$$|P^t f| = (R_{k,0} - L_{0,k})t + O(1). \tag{38}$$

This is a simple corollary of Theorem 2. We see that formula (38) does not contain those left and right rates which are required for formulating the criterion. Naturally, we would like to simplify the criterion, replacing it by something like the following.

Operator P is a wash-out operator if and only if $L_{0,n} > R_{n,0}$.

The criterion we have formulated is valid in one direction (the reverse one); specifically, if $L_{0,n} > R_{n,0}$, then P is a wash-out operator. In the forward direction, however, it is invalid, and a counterexample exists even for n = 2.

Counterexample. Assume that operator P is specified as follows. There are a total of three states: 0, 1, 2. The diameter of the neighborhood is D = 3. The operator acts as shown in Table 1. It is easy to show that $v_{1,0} = -1$, $v_{0,1} = 0$, $v_{2,0} = 1/2$, $v_{1,2} = 1$, $v_{2,1} = 1$. Therefore, pairs (1,0), (0,1), (2,0), (1,2), (2,1) are coupled; (0,2) is not coupled. Indeed,

$$(\ldots.0002222\ldots..)$$
$$\downarrow$$
$$(\ldots.0001222\ldots..)$$
$$\downarrow$$
$$(\ldots.0001122\ldots..)$$
$$\downarrow$$
$$(\ldots.0001112\ldots..),$$

and therefore $L_{0,2} = v_{0,1} = 0, R_{0,2} = v_{1,2} = 1$.

We see that $0 = L_{0,2} < R_{2,0} = 1/2$, and if the new "criterion" were valid, then P would be a non-wash-out operator.

But the operator acts as follows on island $h = (\ldots 000\ \underbrace{111\ldots11}_{l_1}\ \underbrace{22\ldots22}_{l_2}\ \underbrace{11\ldots11}_{l_3}\ 00\ldots)$ with files of ones and twos whose lengths are, respectively, equal to l_1, l_2, l_3.

1) The right file of ones disappears over a time $t_1 = l_3/2$, the left file of ones increases to a length $l_1 + (l_3/2)$, while the length of the twos file remains as before (only shifted to the right by an amount $l_3/2$).

2) After an additional time $t_2 = 2l_2$ the twos file disappears, while the length of the remaining ones file will be $l_1 + 2l_2 + (l_3/2)$.

3) Over an additional time $t_3 = l_1 + 2l_2 + (l_3/2)$ no ones file will remain and the original island is washed out.

Thus, island h is washed out, and since any other island is majorized by an island of the form h, it will also be washed out. Therefore P is a wash-out operator, and the "criterion" is invalid.

5. Evidently, the assertions of items a), b), and c) of Theorem 2 remain valid, if we replace the words "ladder of type (a, b)" in the formulation of Theorem 2 by "state with a finite number of files," but the author could not obtain a proof of this assertion.

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LITERATURE CITED

1. G. A. Gal'perin, "Complexity of computing rates for one-dimensional automaton networks" (in press).
2. A. W. Burks (editor), Essays on Cellular Automata, University of Illinois Press, Urbana (1970).
3. Cellular Structures: Analysis, Synthesis, Behavior [in Russian], Énergiya, Moscow (1973).
4. B. P. Demidovich, Handbook of Problems and Exercises in Mathematical Analysis [in Russian], Nauka, Moscow (1969).
5. A. L. Toom, "Monotonic binary cellular automata," Probl. Peredachi Inform., 12, No. 1, 48-54 (1976).
6. A. L. Toom and L. G. Mityushin, "Two results regarding noncomputability for one-dimensional cellular automata," Probl. Peredachi Inform., 12, No. 2, 69-75 (1976).