Word Problems in Russian Mathematical Education

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In the last few years alarming reports of the declining performance in mathematics of Swedish school children have repeatedly made the news. This has resulted in much hand-wringing and costly and elaborate schemes to counter the trend. No one has the right to claim to have a simple solution to the problem, in fact human communication and learning is a complicated business which is very poorly understood. In the absence of any 'scientific' solution to the problem, a natural thing to do is to take a common-sense approach and look for successful traditions. We believe that such a tradition exists, literally in front of our noses, namely the Russian tradition.

Russian mathematics is highly renowned, and now, after the collapse of the Soviet Union, Russian mathematicians compete for positions around the globe. Meanwhile Russia continues to produce talented young mathematicians. It makes sense to discern and study those features of Russian mathematical education which are responsible for this success. One of us (A.T.) is Russian by origin, the other (U.P.) is Swede. We decided to write this article together because we have a common high opinion about at least one very valuable feature of Russian public education: abundance and variability of word problems at all school levels.

In Russia children begin to solve multistep word problems years before they start to study algebra. They solve these problems by “bare hands”, barely having mastered the basic arithmetical operations, and this is why we call these problems arithmetical. It is a great merit of Russian educators that for many decades they preserved word problems in the curriculum and successfully resisted those delusions which plagued other countries, especially United States. (For a recent relevant comment consult **http://www.insidehighered.com/news/2005/10/10/seniors** or the text in the appendix.)

This Russian tradition can serve as an anti-dote to various so-called progressive methods, whose fashionability threatens to destroy Western mathematical education. Irresponsible fads have already caused much harm in the United States and are on their way to do so in Sweden along with many other Western countries. Much metaphysical speculation on mathematical education has been published, some of it interesting as philosophical reflection. We, on the other hand, shall concentrate on concrete problems and discuss some of them because it is our conviction that the devil is in the details and that discussion of concrete problems combined with keeping alive a succesful tradition will have the most tangible practical consequences for the quality of education.

Following the Russian tradition, and which we believe was once a more or less universal tradition, arithmetics can be subdivided into two parts. First there is the 'mechanical' part, whose object is to familiarize the students with the skills of computing. This is done through so-called exercises (Sw. mekaniska räkneövningar). Dextrity in arithmetical operations needs prolonged instruction and practice, incidentally not unlike the acquisition of muscular co-ordination. This training is necessary although pupils usually do not find it exciting. On the other hand, like with all skills mastered, it inspires pride and satisfaction. One fad in this respect was and still is to side-step the issue by using calculators. Some people claim that the skill of mental computations, as well as by paper and pen, is obsolete because they can be done much faster and more accurately by calculator. There have been plenty of promises that by relieving pupils of the pain of calculation, we allow them to concentrate on the conceptual aspects of mathematics. The arguments pro and contra calculators have already volleyed back and forth for decades and there is no consensus. We do not intend to enter the fray of controversy, only noting that the ulterior object of calculation in school is not getting the answer; it is familiarization with numbers.

The main concern of this article is with the second aspect of elementary mathematical education, namely solving problems as opposed to exercises. The main purpose of solving problems is not to test or train computationsl skills but to stimulate thinking. If a pupil is attracted to mathematics, it is usually through the intellectual challenges it provides. But in order for this to work, at least some rudimentary skill in computation is required. The meat of our article consists in a short list of word problems that have been presented to Russian school children. Those are taken from several sources including the modern books [M] and [G], extensively used in the Russian schools but not, except for a few samples, translated into any other language.

All natural learning occurs in stages, a previous stage being a prerequisite for a subsequent one. To identify stages and their ordering is of course a huge problem, largely unsolved till now. The most famous attempt at a solution has been provided by Piaget. His well-known series of stages, like alternative theories, has subsequently been criticised and remains highly controversial, which is understandable because pedagogy is not an exact science after all. This lack of certainty has inspired some modern pedagogues to claim (in true post-modernistic spirit) that all approaches are equally legitimate. Leaving this controversy aside, we will limit ourselves to state the obvious, namely that every formal instruction necessarily follows some sort of hierarchy, and that structured learning seems particularly essential in elementary mathematics.

One of the most evident structures to follow is to classify problems according to the number of arithmetical operations or “steps” that are necessary to solve each problem. Thus we will speak about one-step problems, two-step problems up to five-step problems. In Russian schools one-step problems normally start to be used at the end of the first grade, two-step problems at the end of the second grade etc. Of course, this classification is necessarily vague, the identification of individual steps is not canonical, and besides the true difficulty of all reasoning (as noted by e.g. William James) is to make the appropriate simplifications. Still in view of the elementary nature of the problems, this inherent ambiguity is not fatal and we believe that the approach is objective enough to be useful.

The sample of word problems presented below is small by necessity. However, we should like to emphasize that the educational value of word problems in Russian schools is based on their large quantity. It is this quantity that allows to include so many different problems that it becomes practically impossible to reduce all problems to a few types and use mechanical application of a few rigid schemes instead of creative thinking. In other words, word problems in Russia are expected to be solved not by methods explained in advance, but *ad hoc* using nothing but bare hands and common sense. The mathematical prerequisite does not include any established theory, only familiarity with numbers and the basic arithmetical operations. Of course, general methods come to the stage, but as a result of solving problems, not as a prerequisite of it. Whenever a pupil solves a problem independently, he grows. Not only he gets a deepened insight into mathematical concepts, but practices logical and clear thinking, and experiences intellectual satisfaction, which strengthens self-confidence and arouses further curiosity. A pupil that fails to solve a problem and needs the solution to be explained, naturally gains much less, yet even in this case his effort has not been entirely in vain if for no other reason than understanding the solution, and even if no progress whatsoever has been made, the solution that is served may provide an inspiration for future efforts.

The worst that can be done with word problems (and often IS done, regretfully) is to consider them as dressed up exercises, the game consisting of guessing what procedure to take, thus stripping them of everything but the numbers and looking for clues as to whether to add or multiply them. We believe that the most charming feature of word problems is the gap between solution of a problem and the numerical procedures used in this solution. Especially elegant are those problems, in which this gap is the greatest since no numbers occur at all, e.g. the problem 9 in this article. Of course, solving problems serves no purpose unless the pupils want to understand and solve them rather than just to satisfy the teacher or to get a grade. It should be also noted that the problems used in Russian schools, including those presented below, are not practical in the sense of necessarily pertaining to the everyday life of the pupils, a notion that has been unduly stressed in much modern pedagogy. Some are evidently fantastic, some involve literary characters (including Pippi Longstocking much beloved by Russian children). Even problems, that may seem connected with everyday life, have a jocular flavour. However, their answers are always determined by the data. All these problems are completely understandable for children and are bound to stimulate the imagination. They are phrased in ordinary language, which encourages children to apply their common sense. Children are generally interested in puzzles, whether they have practical implications or not. It is our belief that solving word problems is one of the most fruitful ways by which children's minds develop.

The Russian experience shows that one does not need to be mathematically gifted to solve such problems. Most intellectually normal pupils can solve them, which prepares them for the next level of abstraction, namely symbolic manipulations, as in algebra. Solving them requires nothing but clear thinking and common sense. The Russian mathematical tradition expects this of all pupils and they meet this expectation.

The Problems

Most of the problems are from two most used in Russia series of textbooks for elementary school: those written by Moro with coauthors and those written by Geidman with coauthors. All these books are carefully edited. Moro's textbooks are the most widely used in Russia today because they are comparatively conservative, similar to what the teachers have got used. Geidman's textbooks are more advanced and used typically by more ambitious teachers. We were told that about 80% of Moscow schools use Moro's and about 10% use Geidman's. However, both series have the the same main feature as those used in Russia for many decades, namely abundance and variability of word problems. The most visible difference between new and old Russian textbooks is that today's books are more attractive, with coloured pictures and Geidman's books have more jocular features.

Problem 1 *Vintik and Shpuntik agreed to go to the fifth car of a train. However, Vintik went to the fifth car from the beginning, but Shpuntik went to the fifth car from the end. How many cars does the train need to have for the two friends to get to one and the same car?*

Problem 2 *Igor and his two friends played chess. Everyone played two games. How many games were played?*

Problem 3 *All the numbers from 1 to 99 were written one after another. How many times was the digit 5 written?*

Problem 4 *Two witches argued, what is faster: mortar or broomstick. They flied one and the same distance of 288 km. The witch in mortar made it in 4 h and the witch on broomstick made it in 3 h. What is greater, speed of mortar or speed of broomstick and how much?*

Problem 5 *Villa "Chicken" occupies a rectangle 26 m wide and 40 m long. The house occupies 1/4 of the area and 1/5 of the area is occupied by flowers. The remaining part is occupied by trees. How large is the area occupied by trees?*

Problem 6 *When it started to rain, Pippi put an empty barrel with 400 liters of capacity under a drain-pipe. Every minute 8 liters of water enter the barrel and 3 liters go out of it through chinks. Did the barrel get filled if it was raining for 1 hour 10 minutes?*

Problem 7 *A plane has two gasoline tanks. The total amount of gasoline in both tanks is 24 liters. The first tank contains 4 liters of gasoline more than the second. How much gasoline is there in each tank?*

Problem 8 *A boy and a girl collected 24 nuts. The boy collected two times more nuts than the girl. How many did each collect?*

Problem 9 *When Ivan Tsarevich came to the Magic Kingdom, Koschey was as old as Baba Yaga and Ivan Tsarevich together. How old was Ivan Tsarevich when Koschey was as old as Baba Yaga was when Ivan Tsarevich came to the Magic Kingdom?*

Ivan Tzarevich, Koschey and Baba Yaga are well-known characters of Russian folk tales.

Problem 10 *A library needs to bind 4500 books. One shop can bind these books in 30 days, another shop can do it in 45 days. How many days are necessary to bind all the books if both shops work at the same time?*

Problem 11 *Deniska can eat a jar of jam in 6 minutes. Mishka can eat a similar jar of jam two times faster. In how much time will they eat a jar of jam together?*

Deniska and Mishka are colloquial versions of common Russian names, which fit into the jocular style of this problem.

Problem 12 *A boat, going downstream, made a distance between two ports in 6 hours and returned in 8 hours. How much time is needed for a raft to make this distance downstream?*

Problem 13 *An ancient problem. A flying goose met a flock of* *geese in the air and said: "Hello, hundred geese!" The leader of the flock answered to him: "There is not a hundred of us. If there were as many of us as there are and as many more and half as many more and quarter as many more and you, goose, also flied with us, then there would be hundred of us." How many geese were there in the flock?*

Problem 14 *Three friends played chess so that every two of them played one and the same number of matches. After that they argued who is the winner. The first one said: I won more matches than each of you. The second one said: I lost less matches than each of you. The third player said nothing, but when the points were counted, they found that he had gained more points than each of the others. Is it possible? (A victory brings a point, a draw brings half a point, a loss brings nothing.)*

Problem 15 *A swimmer was swimming upstream Neva River. Near the Republican Bridge he lost an empty flask. After swimming 20 min more upstream he noticed his loss and swam back to find the flask; he reached it near the Leughtenant Schmidt Bridge. Find the velocity of current at Neva River if the distance between these two bridges is 2 km.*

Problem 16 *Two old women started at sunrise and each walked at a constant velocity. One went from A to B and the other from B to*

A. They met at noon and, continuing with no stop, arrived respectively at B at 4 p.m. and at A at 9 p.m. At what time was the sunrise on that day?

Problem 17 *A team of mowers had to mow two meadows, one twice as large as the other. The team spent half-a-day mowing the bigger meadow. After that the team split. One half of it remained at the big meadow and finished it by the evening. The other half worked on the smaller meadow, but did not finish it at that day. The remaining part was mowed by one mower in one day. How many mowers were there?*

Problem 18 *A man sold firewood. To make standard portions, he always used one and the same rope, surrounded a pack of logs with it and brought it into houses on his back. One woman asked him to bring a double portion of firewood. The man proceeded as usual, only took a rope one and a half times longer than usual. The woman complained: "Since I payed you a double fee, you should use a double length rope." The man replied: "You are mistaken, mam. In fact, I brought you even a little more firewood than you requested." Who is right?*

Problem 19 $2n$ *knights came to King Arthur's court, each having not more than* $n-1$ *enemies among the others. Prove that Merlin (Arthur's advisor) can place the knights at a round table in such a man-*

ner that nobody will sit beside his enemy.

Comments on Problems

In elementary school Russian pupils are required to present solution of every word problem as a series of questions answered by an arithmetical operation. For example, a model solution of problem **10** may look as follows:

How many books can the 1-st shop bind in 1 day?

$$4500/30 = 150$$

How many books can the 2-d shop bind in 1 day?

$$4500/45 = 100$$

How many books can the two shop bind in 1 day?

$$150 + 100 = 250$$

In how many days can the two shops bind the books?

$$4500/250 = 18$$

Answer: the two shops can bind the books in 18 days.

Counting steps, we see that it is a four-step problem. Problem **10** may be called a "forward problem" because it can be solved in a straightforward way, just performing several arithmetical operations, every single one having an evident meaning. (But it does not mean that the problem as a whole is evident! For many children it is not.) However simple, forward problems are an indispensable stage of development of every child's mathematical competence. This stage, in its turn, consists of several stages with growing number of steps in their solutions and every child should pass this ladder step by step.

Now let us try to classify non-forward problems. Some of them come into a category "Find two numbers given their sum and difference". The problem **7** obviously belongs here. These problems seem suitable for the third or fourth grade. Of course, this type of problems (as any type) should not be overdone. As soon as a problem has been spotted as belonging to a certain type, it does not provide a challenge any more unless it comes as a part of a more ambitious problem.

Another category of non-forward problems is "Find two numbers given their sum and ratio". In Russia such problems have a special name "problems on parts" because you can solve them without algebra by introducing "parts". These problems are so common in Russia that a well-known writer, Nosov, described one of them in his book "Vitya Maleev at school and at home". The hero, Vitya Maleev, has just finished the third grade. He failed in mathematics and promised his teacher to train himself in solving problems to catch up. So he tries to solve the problem **8** (from the 3-grade textbook used at that time, that is in 1952).

First Vitya divides 24 by two and gets 12. May be, each collected 12 nuts? No, the boy collected more than the girl. Not knowing what to do, Vitya draws a picture of a boy and a girl. To express the fact that the boy collected two times more nuts than the girl, Vitya draws two pockets on the boy's pants and one pocket on the girl's apron. Then he looks at his picture and sees **three pockets**. *Then an idea "like a lightning" comes to his mind: the nuts should be put into these pockets, so he should divide the number of nuts into the number of pockets! Thus he gets 24 : 3 = 8. So each pocket contains 8 nuts. This is how many nuts the girl has. The boy has two pockets, so he has 8 times 2, which gives 16. Now Vitya can check his answer: he adds 8 and 16 and gets 24. Now he is sure that his solution is correct. He is very excited. Probably, this is the first time in his life that he solved a problem on his own. He goes to the street to tell somebody about it. A neighbor girl says: "This is a third-grade problem. We solved them last year." This does not diminish Vitya's joy and he is right: he made a discovery.* In United States this problem (as well as other problems of our sample) would be called *bogus* or *phony* because in everydy life those characters would simply count the nuts. What about Vitya, such a student probably would be put into the "consumer math" track and remain a mathophobe for the rest of his life.

The solution presented above shows the importance of visualization in problem solving. This is not surprising as many ideas cannot be properly articulated and often present themselves visually. Another simple example of this is problem **1** where an appropriate diagram makes the whole thing obvious.

There are many other types of problems. More than that, different classifications of problem may intersect each other in various ways. Every single type is specific, but every one is useful for children as a stepping stone on their path of development and maturation. Ideally, every child at a certain age should comprehend how to solve problems of a sufficient number of different types appropriate for this age, and the more independently the better.

An elaboration of the "parts" type, suitable for grades 6-8 (12-14 years old) is problem **13**. It is especially useful to solve first by parts, without algebra, and then to solve again, using algebra.

Now let us speak about more complicated problems. We shall see that they also form certain types. The problem **12** appeared first in a book by Ya. I. Perelman, a famous Russian educator, then at a Moscow mathematical olympiad in 1940 and a few years later was included into a school problem book for 5-6 grades! It is more difficult than the previous problems and one may even suggest that the problem is unsolvable and we should not disparage this idea because unsolvable problems **do exist** and children should be alert to meet them; remember the Captain's Age phenomenon. So the fact that problem **12** is after all solvable is non-trivial and may lead to deep reflexions. One way to go around its seeming unsolvability is to introduce an arbitrary distance between the two ports, say 48 miles. The resulting problem is forward: we easily find that going down the current, the boat makes 8 miles per hour, while going against it, it only manages 6 miles per hour. Why are these velocities different? Evidently, because of the current: in one case it increases the velocity of the boat, in the other case it decreases it. Thus velocity of the current equals half of the difference of the two velocities, that is 1 mile per hour. (Notice that the type "find two numbers given their sum and difference" is assumed to be thoroughly comprehended and mastered, which gives an example of one skill built on another.) The raft's speed equals the current's speed, namely 1 mile per hour, so it will take 48 hours for the raft to make the travel. Now we can take another distance, say 24 miles, and get the same answer! This suggests that the answer is independent of the distance, so the problem can be solved without knowing it. To complete the solution, one needs somehow to confirm this suggestion, for example introducing a special unit of length equal to the distance between the ports.

The problem **10** will come to this type also if the total number of books is not given. Indeed, answers in both problems do not depend on the number of books in one case and the distance in the other. Problem **11** also belongs to this type, but it has smaller numbers, which makes it more accessible. The pupils may observe that in six minutes the two gluttons eat three jars of jam together, hence they should be able to finish one in two minutes. Note that due to the simplicity there is no need for the child ever to have encountered fractions let alone being taught how to add them.

Now let us make some general comments. These problems deal with rates, be it ordinary velocities or rate of book-binding or jam-eating. The crucial insight is that rates are additive. Of course, this is a simplification. One may argue that book-binding does not follow such a steady rate, because when two book-binders start to co-operate, they may get in each other's way, (as well as the two jam-eating boys might have their spoons clash) and rates would not add up. (And of course we know that for high speeds, by special relativity theory, velocities do not add.) Such concerns might be valid in a real-life application, but the point is that these problems are not meant to be real-life applications. They are playful in spirit. Children easily get this spirit, but some grown-ups remain deaf for it, especially some of those who have a degree in education. Mathematical and scientific thinking always involves simplification, it strips situations of the extraneous. This ability to strip down is essential in all problem solving as it is in all kinds of reasoning (see [WJ], especially pp. 329-360).

As a somewhat more elaborate problem involving addition of rates we may consider **15**. This problem shows the power of the physical idea of relative movement. Let us place ourselves in the coordinate system moving with the stream. In this system the flask does not move when it is lost, while the swimmer swims first away from it, then towards it. His proper velocity is assumed constant, so he spends one and the same time swimming in both directions. But one of them took 20 minutes, so the other also takes 20 minutes, so the total time when the flask was lost is 40 minutes. Now we return to the coordinate system where we were before, and notice that the flask took 40 minutes to move from one bridge to the other, that is 2 km. So the velocity of the current is 2 km divided by 2/3 hour, which makes 3 km/h.

Now let us go to the most involved problems in our sample. Let us start with the problem **16**. To solve it, one may draw a diagram with distance from A and time as coordinates and use similarity of triangles. Vladimir Arnold, a famous Russian mathematician, emphasizes that solving this problem independently when he was 12 years old was his first real mathematical experience, and he uses the words *revelation* and *feeling of discovery* to describe this experience and says that in Russia his experience was not unusual.

The problem **17** comes from the nineteenth century and it is said that Leo Tolstoi, the famous writer who was very interested in public education, liked it. It also can be solved without algebra. A visual representation, as with so many other problems, can be helpful.

Now let us turn to geometric word problems. First about the problem **18**. To solve this problem, we, as usual, have to make simplifying assumptions. We assume that the firewood surrounded by a rope is a cylinder, whose height is the length of the logs and base's circumference equals the lenght of the rope. Since the height of the cylinder is constant, the volume of the cylinder is proportional to the area of the base, which is proportional to the square of the radius or, which is the same, to the square of the circumference. So, if the length of the rope is multiplied by 3/2, the amount of the firewood is multiplied by a square of this amount, which is 9/4, which is really a little more than 2. The man was right.

Going to more difficult problems, we naturally go from school to mathematical olympiads. In Russia there is no gap between them: harder school problems come close to olympiad level. The problem **12** is an example of this proximity.

On the other side, olympiad problems border on university mathematics. For example, if you want to introduce children into graph theory, you don't need to start with cumbersome terminology and definitions. Instead you can give them a problem like **19**. This problem was proposed for the 27th Moscow Mathematical Olympiad, but it was unusable in its original form: *A graph has* $2n$ *vertices, each vertex being incident with at least* n *edges. Prove that this graph has a hamiltonian cycle.* One of us (A.T.) proposed to represent the hamiltonian cycle by the legendary round table, and in this form the problem was accepted. After that it was discussed at mathematical circles, where knights were represented by circles and friendship relations by lines connecting them. Thus discussion of a "jocular" problem smoothly turned into a study of graph theory, which was non-trivial from the very beginning.

Bibliography

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[M] M.I. Moro et al *Mathematics. Textbook for the fourth grade of elementary school in two parts. Part 2 (Second half-year).* "Prosveshenie", Moscow, 2004. (In Russian.)

[WJ] W.James *Principles of Psychology II* Dover Inc, 1890 (1950)

Appendix.

A recent document from USA.

Abstract from the NASSMC Briefing Service (NBS) that is supported in part by the International Technology Education Association and Triangle Coalition for Science and Technology Education, Monday, October 17, 2005. Original article appeared in Inside Higher Ed., Monday, October 10, 2005

> *A national survey of the high school class of 2004 shows that, while today's students are fairly ambitious in their expectations for post secondary education, they are under-prepared for college mathematics.*
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> *The survey by the National Center for Education Statistics shows that 62 percent of students expected to go to a four-year college. Thirty-five percent expected to pursue a graduate degree. Among racial groups, Asian students were the most optimistic, and Latino and American Indian students the least.*
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> *However, a distressing proportion of the students who had high aspirations proved minimally prepared in mathematics. Only half of those who expected to earn a graduate degree had an "understanding of intermediate-level mathematical con-*

cepts" or the "ability to formulate multistep solutions to word problems."

One-third hadn't mastered "simple problem solving, requiring the understanding of low-level mathematics concepts."