On Critical Values for Some Random Processes with Local Interaction in $\mathbf{R}^2$

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ABSTRACT We complete the proof of a necessary and sufficient condition for existence of non-trivial critical values for some classes of random processes with local interaction, where the space is a real plane. Our operators are superpositions of a deterministic operator and a one-sided random noise, where the noise is standard and the geometric properties of the deterministic operator are crucial.

The problem of ergodicity for random processes with local interaction deserves attention for various reasons. These processes are useful to model many natural phenomena, and ergodicity or lack of it is one of their basic properties. It is well known that the problem of ergodicity for random cellular automata is algorithmically unsolvable, even in the one-dimensional discrete case [1]. So we should look for special classes of processes, for which we can present criteria of ergodicity. One such case was studied in [2]. Let us formulate the corresponding result in a way appropriate here. The space in this case is $\mathbb{Z}^2$ and the configuration space Ω is the set of all subsets of $\mathbb{Z}^2$. Let us consider deterministic operators $D : \Omega \to \Omega$, which are subject to the following conditions, where DS means the result of application of operator D to the set $S \subseteq \mathbb{Z}^2$:

a) *Locality:* there is a finite *range of interaction* r such that whether the point $(i, j) \in \mathbb{Z}^2$ belongs to DS depends only on the set

$$\{(a, b) \in S : |a - i| + |b - j| \leq r\}.$$

b) *Uniformity:* D commutates with all shifts of Ω induced by shifts of the space $\mathbb{Z}^2$.

c) *Monotonicity:* if $S \subset S'$, then $DS \subseteq DS'$.

d) *Non-triviality:* D does not map all sets to one and the same set.

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It is easy to prove that for any operator D satisfying these conditions there is a non-empty finite family Ψ of finite subsets of $\mathbb{Z}^2$, such that the action of D can be presented as

$$DS = \bigcap_{\psi \in \Psi} (S + \psi), \tag{1}$$

where + means vector summation: $A + B = \{a + b,\ a \in A,\ b \in B\}$. In [2, 3] analogs of elements of Ψ were called zero-sets. We call D an *eroder* if for any bounded set $S \subset \mathbb{Z}^2$ there is t such that $D^t S = \emptyset$. If D is an eroder, the empty set is similar to a state of stable equilibrium in physical systems, which is one reason to study eroders. It is interesting that to study them even in the discrete case we need to imbed $\mathbb{Z}^2$ into a real space $\mathbb{R}^2$, where we define the set $\sigma \subset \mathbb{R}^2$ as the intersection of convex hulls of all $\psi \in \Psi$. Theorem 1 (below) shows that σ is relevant to behavior of deterministic and random operators and thereby is an example of the importance of geometry of interaction for the study of ergodicity.

Now let us define a random operator G_ε, where $\varepsilon \in [0, 1]$ is a parameter. By definition, G_ε turns any set $S \subseteq \mathbb{Z}^2$ into a random set $S \cup B$, where B is a random set, which includes any element of $\mathbb{Z}^2$ with a probability ε independently of other elements. Let us iteratively apply the superposition $G_\varepsilon D$ (where D is applied first, G_ε second) to the empty set as the initial condition. Of course, every application of $G_\varepsilon D$ involves generation of a new random set. Thus we obtain a sequence of random sets

$$(G_\varepsilon D)^t \emptyset. \tag{2}$$

Since all of these random sets are space-uniform, each of them has a *density*, which can be defined as the probability that the origin belongs to this set. The following theorem is a direct consequence of results of [2].

Theorem 1. *For all operators D defined above:*

a) *If σ is non-empty, then D is not an eroder and the density of the sets* (2) *tends to 1 when $t \longrightarrow \infty$ for every positive ε.*

b) *If σ is empty, then D is an eroder and the density of the sets* (2) *tends to 1 when $t \longrightarrow \infty$ only for large enough ε. For small enough ε this density tends to a limit, which is less than 1 and tends to zero when $\varepsilon \longrightarrow 0$.*

The main purpose of the present article is to complete the proof of a similar result for continuous space. Now we must be careful with our definitions not to run into unmeasurable sets. There are various ways to avoid this danger and we choose the following. For any $r > 0$ the set $\{p \in \mathbb{R}^2 : |p| \leq r\}$, where $|\cdot|$ is the Euclidean norm, will be denoted *Disk* (r). For any non-empty family Ψ of closed subsets of *Disk* (1) we define a deterministic operator D_Ψ by the following rule: for any closed $S \subseteq \mathbb{R}^2$ the result of application of D_Ψ to S is defined as

$$D_\Psi \, S = \bigcap_{\psi \in \Psi} (S + \psi), \tag{3}$$

where plus means vector sum of sets in $\mathbb{R}^2$ as a vector space. Elements of Ψ are analogs of elements of Ψ in (1), but now they are subsets of a continuous space and the set Ψ may be infinite. Since all elements of Ψ are closed, D_Ψ transforms any closed set into a closed set; this is important to be sure that all sets (4) are measurable.

The random operator also has to be defined carefully now not to run into unmeasurable sets. For this reason we use a very specific random growth operator $G_{\varepsilon,\, d,\, r}$, which turns any closed set $S \subseteq \mathbb{R}^2$ into a random set $S \cup \Gamma_{\varepsilon,\, d,\, r}$, where $\Gamma_{\varepsilon,\, d,\, r}$, which we call the *growth set*, is a random set defined as follows. First, for any $d > 0$ we denote $\mathbb{Z}_d$ the set $\{k \cdot d : k \in \mathbb{Z}\}$. We choose an orthogonal coordinate system in our plane and for any positive d denote $\mathbb{Z}_d^2$ the set of points, both of whose coordinates x, y belong to $\mathbb{Z}_d$. The growth set is the union of closed disks with radii r and centers at the points belonging to $\mathbb{Z}_d^2$, each taken with probability ε independently of others. Notice that $\Gamma_{\varepsilon,\, d,\, r}$ is closed a.s. We shall always take $r \geq d/\sqrt{2}$ because otherwise even all of our disks together do not cover the plane. We are interested in the behavior of random sets

$$(G_{\varepsilon,\, d,\, r} \, D_\Psi)^t \, \emptyset \tag{4}$$

resulting from t applications of the composition $G_{\varepsilon,\, d,\, r} \; D_\Psi$ (first D_Ψ, then $G_{\varepsilon,\, d,\, r}$) to the initial condition $\emptyset$, which means the degenerate random set concentrated in the empty set. (We hope that it will cause no confusion if we denote any random set concentrated in one set in the same way as the set itself.) Of course, every application of $G_{\varepsilon,\, d,\, r}$ involves generation of a new random set $\Gamma_{\varepsilon,\, d,\, r}$. Notice that the set (4) is closed a.s. and that its intersection with any disk is a finite linear combination of degenerate random sets, everyone of which is concentrated in one set. This assures that all our definitions make sense. We call *density* at time t the limit

$$\mathit{Density}\ (t) = \lim_{q \to \infty} \frac{1}{\pi q^2} \, E|\mathit{Disk}\ (q) \cap (G_{\varepsilon,\, d,\, r} \, D_\Psi)^t \, \emptyset|, \tag{5}$$

where E is expectation and $| \cdot |$ means measure, which exists a.s. because the set is closed a.s. Existence of the limit is easy to prove. (Unlike the discrete case, we do not speak about a limit of random sets, because in the continuous case is is not so easy to define.)

Our purpose is to extend Theorem 1 to the continuous case. The literal analog of Theorem 1 for continuous space is false, because in this case an operator may be an eroder even when σ is not empty (see Example 3 below). However, the following definition saves the day: let us call D_Ψ a *linear eroder* if there is a positive constant C such that t applications of D_Ψ turn any bounded set S into an empty set as soon as t exceeds C times the diameter of S. Our main result is the following analog of Theorem 1:

Theorem 2. *For all operators D_Ψ defined by* (3)*:*

a) *If σ is non-empty, then D_Ψ is not a linear eroder and the density of the random sets* (2) *tends to* 1 *when $t \to \infty$ for every positive ε.*

b) *If σ is empty, then D_Ψ is a linear eroder and the density of the random sets* (4) *tends to 1 when $t \to \infty$ only for large enough ε. For small enough ε this density tends to a limit, which is less than 1 and tends to zero when $\varepsilon \to 0$.*

Most of the statements of Theorem 2 were proved in [3] in any dimension. The only statement not proved there was the second part of the statement a): if σ is non-empty, then density of the random sets (4) tends to 1 for any positive ε. The main purpose of this article is to prove this, although only for the 2-dimensional case. A similar result is probably true in all dimensions, but it needs a more elaborate proof. But first let us consider several examples.

Example 1. Let us take an equilateral triangle T with side 1 and center at the origin. The elements of Ψ are triples of points lying one-to-one on all three sides of this triangle. In this case σ is empty, so the operator is an eroder and there is a critical value of ε.

Example 2. Let us take a square Q with side 1 and center at the origin. The elements of Ψ are quadruples of points lying one-to-one on all four sides of this square. In this case σ is non-empty and the operator is not an eroder, so there is no critical value of ε.

Examples 1 and 2 do not yet show anything new by comparison with the discrete case, but example 3 does.

Example 3. Elements of Ψ are closed arcs with center O and radius 1, whose radian measure is π, that is halves of the circumference of *Disk* (1). The set σ is not empty: it consists of one point O. In this case D_Ψ is an eroder, but not a linear eroder: it takes $\asymp R^2$ applications of D_Ψ to turn a disk with a radius R into an empty set. (Generally, for any non-linear eroder the number of applications of D, which are necessary to erode a disk with a radius R, is not less than $\asymp R^2$, but may be much greater for large R.) This example illustrates the main difficulty with which we deal in this article. You may want to keep this example in mind while reading the subsequent general proof of the statement a) of our theorem.

In our proof we use the notions of monotonicity and order. Let us call a real function f on the set of closed subsets of $\mathbb{R}^2$ *local* if $f(S)$ actually depends only on intersection of S with some disk. Let us call a local function f *monotonic* if $S_1 \subset S_2$ implies $f(S_1) \le f(S_2)$. Given a local function f and a random set μ, we denote $E(f \mid \mu)$ the expectation of f according to μ (if it exists). Given two random sets μ_1, μ_2, we write $\mu_1 \prec \mu_2$ if $E(f \mid \mu_1) \leq E(f \mid \mu_2)$ for all monotonic f. We call an operator F acting on random sets *monotonic* if $\mu_1 \prec \mu_2$ implies $F\mu_1 \prec F\mu_2$. It is easy to show that all our operators D_Ψ and $G_{\varepsilon,\,d,\,r}$ are monotonic. Therefore *Density* (t) is a non-decreasing function of t and has a limit when $t \to \infty$. It is easy to prove that for any $d > 0$ and $r > d/\sqrt{2}$ this limit equals 1 provided ε is large enough.

In the same vein as it was done in [3] for block sets, it is easy to prove that for any positive d_1, d_2, $r_1 \geq d_1/\sqrt{2}$, $r_2 \geq d_2/\sqrt{2}$ and ε_1 there is a positive ε_2 such that $\Gamma_{\varepsilon_1,\,d_1,\,r_1} \succ \Gamma_{\varepsilon_2,\,d_2,\,r_2}$ and there is a positive ε_2' such that $\Gamma_{\varepsilon_1,\,d_1,\,r_1} \prec \Gamma_{\varepsilon_2',\,d_2,\,r_2}$. Due to these inequalities, it is sufficient to prove our statements only for some positive values of d and $r \geq d/\sqrt{2}$, which we are free to choose as we like. Hence we fix $d = 0.1$ and $r = 100$ and $G_{\varepsilon,\,d,\,r}$ turns into $G_{\varepsilon,\,0.1,\,100}$, which we abbreviate as G_ε. Since σ is non-empty, we may assume without loss of generality that it contains the origin. Starting here we assume that some D_Ψ such that σ contains O and some $\varepsilon > 0$ are chosen. Our goal is to prove that density defined by (5) tends to 1 when $t \to \infty$. This follows immediately from the following: For any $p \in \mathbb{R}^2$ and any positive q,

$$\lim_{t\to\infty} \text{Prob}\ (p + \textit{Disk}\ (q) \subseteq (G_\varepsilon\ D_\Psi)^t\ \emptyset) = 1. \tag{6}$$

It remains to prove (6). We shall prove it for $p = O$, the general proof is the same. Notice that for any $\varepsilon > 0$ the expression $(1 - (1 - \varepsilon)^n)^n$ tends to 1 when $n \to \infty$. Using this, let us choose the minimal natural n for which $(1 - (1 - \varepsilon)^n)^n \geq 0.99$. Let us define a sequence

$$q_k = 1000\, n \cdot 2^n \cdot 2^k, \text{ where } k = 0, 1, 2, 3, \ldots$$

and prove that for any $q = q_k$,

$$\text{Prob}\ (\textit{Disk}\ (2q)\ \subseteq\ (G_\varepsilon\ D_\Psi)^{100\, q\cdot n\cdot 2^n}\ \ \textit{Disk}\ (q))\ \geq\ 1 - q \cdot e^{-q}. \tag{7}$$

Let us explain why (7) implies (6). The infinite product

$$\prod_{k=k_0}^{\infty} (1 - q_k \cdot e^{-q_k}), \text{ where } k_0 = 0, 1, 2, \ldots .$$

is a lower estimation of probability of unlimited growth of our disk on condition that the initial set included *Disk* (q_{k_0}) at some initial time. It is easy to check that all the factors of this product are positive and that the sum of their logarithms converges, whence the infinite product converges to a positive number which tends to 1 when k_0 tends to ∞. But a disk with any radius has a positive probability to appear as part of (4) at any place at any time. So, at least one of them will grow to infinity a.s. (Actually, the law of zero or one is at work here.)

It remains to prove (7). Let us consider configuration space $\mathbb{R}_+^q$ and a map Π transforming any configuration $a = (a_0, \ldots, a_{q-1})$ in this space into a closed set $\Pi(a) \subset \mathbb{R}^2$ surrounded by the polygon $S_0 \ldots S_{q-1}$ whose vertices S_i are defined by their polar coordinates ϕ angle and ρ radius:

$$\phi(S_i) = \frac{2\pi i}{q} \text{ and } \rho(S_i) = a_i \text{ for } i = 0, \ldots, q-1.$$

To prove (7), it is sufficient to prove that for any $q = q_k$,

$$\text{Prob}\ (\Pi\,(\underbrace{3q, \ldots, 3q}_{q}) \subseteq (G_\varepsilon\, D_\Psi)^{10q\cdot n\cdot 2^n}\, \Pi\,(\underbrace{q, \ldots, q}_{q})) \geq 1 - q \cdot e^{-q}. \quad (8)$$

Let us explain why (8) implies (7). Suppose that we have *Disk* (q), where $q = q_k$, as the initial configuration. Notice that $\Pi(q, \ldots, q)$ is a regular polygon inscribed into the circumference of *Disk* (q). According to (8), $10q \cdot n \cdot 2^n$ applications of $G_\varepsilon\ D_\Psi$ turn $\Pi(q, \ldots, q)$ into a set containing $\Pi(3q, \ldots, 3q)$ with a probability at least $1 - q \cdot e^{-q}$. Since *Disk* $(q) \supset \Pi(q, \ldots, q)$, the same is true of *Disk* (q) from monotonicity. But $\Pi(3q, \ldots, 3q)$ contains *Disk* $(2q)$, whence (7) immediately follows.

It remains to prove (8). Let us consider the following growth process, whose configuration space is $\mathbb{R}^q$. First we define a deterministic operator $\overline{D} : \mathbb{R}^q \to \mathbb{R}^q$ which transforms any $(a_0, \ldots, a_{q-1})$ into $(b_0, \ldots, b_{q-1})$, where

$$b_i = \frac{a_i + \min(a_{i-1},\ a_i,\ a_{i+1})}{2} - \frac{4}{q} \text{ for all } i = 0, \ldots, q-1, \quad (9)$$

where $i-1$ and $i+1$ are modulo q.

We also define a random growth operator $\overline{G}_{\varepsilon,\,\delta}$ which acts on normed measures on $\mathbb{R}^q$ transforming any $a \in \mathbb{R}^q$ into a product measure, induced by i.i.d. random variables $\overline{g}_i$, everyone of which equals δ with probability ε and 0 with probability $1 - \varepsilon$ with the map

$$b_i = \min(a_i + \overline{g}_i,\ 3q). \quad (10)$$

We shall prove for all natural t that

$$(G_\varepsilon\, D_\Psi)^t\, \Pi\,(\underbrace{q, \ldots, q}_{q}) \succ \Pi\,(\overline{G}_{\varepsilon,\,0.1}\, \overline{D})^t\,(\underbrace{q, \ldots, q}_{q}), \quad (11)$$

where $\overline{G}_{\varepsilon,\,0.1}$ is $\overline{G}_{\varepsilon,\,\delta}$ with $\delta = 0.1$. Also we shall prove that

$$\text{Prob}\left(\min_i(a_i) \geq 3q \mid (\overline{G}_{\varepsilon,\,0.1}\, \overline{D})^{100\,q\cdot n\cdot 2^n}\ (\underbrace{q, \ldots, q}_{q})\right) \geq 1 - q \cdot e^{-q}, \quad (12)$$

where $\text{Prob}\ (A \mid \mu)$ means probability of event A according to measure μ.

Before proving (11) and (12), let us explain why they imply (8). Let us consider probabilities of the event "the resulting random set contains $\Pi(3q,\ldots,3q)$" according to both measures in (11), where $t = 100,\ q \cdot n \cdot 2^n$. Since the indicator function of this event is monotonic, (11) implies inequality of the same sense between these probabilities:

$$\mathrm{Prob}\left(\Pi(\underbrace{3q,\ldots,3q}_{q}) \subseteq (G_\varepsilon\, D_\Psi)^{100\,q\cdot n\cdot 2^n}\,\Pi\,(\underbrace{q,\ldots,q}_{q})\right) \tag{13}$$

$$\geq \mathrm{Prob}\left(\Pi(\underbrace{3q,\ldots,3q}_{q}) \subseteq \Pi\,(\overline{G}_{\varepsilon,\,0.1}\,\overline{D})^{100\,q\cdot n\cdot 2^n}\,(\underbrace{q,\ldots,q}_{q})\right).$$

From (12), the latter probability in (14) is not less than $1 - q \cdot e^{-q}$, so the former probability also is not less than the same number, which amounts to (8).

Now let us prove (11) by induction using the inequality

$$G_\varepsilon\, D_\Psi\, \Pi \succ \Pi\, \overline{G}_{\varepsilon,\,0.1}\, \overline{D} \tag{14}$$

as the induction step. It is sufficient to prove (14) only when both parts are applied to configurations, all the components of which are between $q/2$ and $3q$. Let us explain why. According to (9), the minimum of components can decrease at most by $4/q$ at every application of $\overline{D}$ and cannot decrease at all when $\overline{G}_{\varepsilon,\,\delta}$ is applied. Therefore, in the course of $100\,q \cdot n \cdot 2^n$ applications of $\overline{G}_{\varepsilon,\,\delta}\,\overline{D}$ the minimum of components can decrease at most by $400\,n{\cdot}2^n$, so all the components of the resulting configurations will be not less than $q - 400\,n \cdot 2^n > q/2$ at all times from 0 to $100\,q \cdot n \cdot 2^n$. The maximum of components will never exceed $3q$ due to (10).

In its turn, (14) follows from these two inequalities, where both sides are applied to elements of $[q/2,\,3q]^q$: 1) $D_\Psi\,\Pi \succ \Pi\,\overline{D}$ and 2) $G_\varepsilon\,\Pi \succ \Pi\,\overline{G}_{\varepsilon,\,0.1}$.

Let us prove 1). Since $O \in \sigma$, the result of application of D_Ψ to any closed half-plane contains this half-plane. Let us observe that in our process $|a_{i,\,t} - a_{i+1,\,t}| < 0.2$ a.s., where $i+1$ is modulo q. Lengths of sides of our polygon are the smallest if $a_i \equiv q/2$ and in this case they are $q\sin(\pi/q)$, which is greater than 3. The biggest length of its side does not exceed 30, because the radius cannot be greater than $3q$ and $|a_i - a_{i+1}|$ is less than 0.2. Also observe that angles of our polygon are not less than $\pi/2$. Due to all this, we can represent the difference between $\Pi(a)$ and the result of application of D_Ψ to it as

$$\Pi(a) \setminus D_\Psi\,\Pi(a) \subseteq \bigcup_{\alpha_i<\pi} T_i,$$

where $\setminus$ means difference of sets, α_i is the radian measure of the angle $S_{i-1}\,S_i\,S_{i+1}$ and T_i is some figure in the vicinity of S_i, which we are going to examine. Let us concentrate on T_0. The orthogonal coordinates of the three relevant points are

$$S_0 = (a_0,\ 0),$$
$$S_1 = (a_1 \cos(2\pi/q),\ a_1 \sin(2\pi/q)),$$
$$S_{-1} = (a_{-1} \cos(2\pi/q),\ -a_{-1} \sin(2\pi/q)).$$

It is sufficient to look at the case $a_0 > \min(a_{-1}, a_0, a_1)$. In this case, as we have seen, $a_0 < \min(a_{-1}, a_0, a_1) + 0.2$ and from monotonicity we may assume that $a_1 = a_{-1} = a_0 - \Delta$, where $0 \le \Delta < 0.2$. Let us draw a line parallel to the y axis such that the distance between its points of intersection with $S_0 S_1$ and $S_0 S_{-1}$ equals 2. The distance of S_0 from this line is an upper estimation of the amount by which a_1 decreases. It is easy to calculate that this distance equals

$$\frac{a_0}{a_0 - \Delta} \tan \frac{\pi}{q} \ + \ \frac{\Delta}{a_0 - \Delta} \cot \frac{2\pi}{q}.$$

The former addend does not exceed $4/q$ and the latter addend does not exceed $\Delta/2$. Thus 1) is proved. What about 2)? It is true because $G_\varepsilon = G_{\varepsilon,\, d,\, r}$ where $d = 0.1$ is small enough and $r = 100$ is large enough. Thus (11) is proved.

It remains to prove (12). Let us consider another random growth process whose configuration space is R^m, where $m = q/n$ and a generic configuration is $a = (a_0, \ldots, a_{m-1})$. We consider two operators acting on measures on R^m. The first operator, called $\overline{\overline{D}}$, is deterministic. It transforms any configuration a into b defined by

$$b_i \ = \ \min(a_{i-1},\ a_i,\ a_{i+1}) - \frac{4}{m}, \tag{15}$$

where $i-1$ and $i+1$ are modulo m.

Another operator $\overline{\overline{G}}_{\beta,\,\gamma}$ transforms any configuration $a \in \mathbb{R}^m$ into a product measure induced by i.i.d. random variables $\overline{\overline{g}}_i$, everyone of which equals γ with probability β and 0 with probability $1-\beta$ with the map

$$b_i \ = \ \min(a_i + \overline{\overline{g}}_i,\ 3q). \tag{16}$$

Let us define a deterministic operator $Q\ :\ \mathbb{R}^q \to \mathbb{R}^m$ by the rule

$$(Qa)_i \ = \min_{ni \le j < n(i+1)} a_j.$$

We shall prove that

$$Q\ (\overline{G}_{\varepsilon,\,0.1}\ \overline{D})^{nt}\ (\underbrace{q, \ldots, q}_{q})\ \succ\ (\overline{\overline{G}}_{\beta,\,\gamma}\ \overline{\overline{D}})^t\ Q\ (\underbrace{q, \ldots, q}_{q}) \tag{17}$$

for all natural t, where

$$\beta \ = \ (1-(1-\varepsilon)^n)^n \ \ge \ 0.99 \text{ and } \gamma \ = \ 0.1 \cdot (1/2)^n. \tag{18}$$

We shall also prove that

$$\text{Prob}\left(\min_i(a_i) \geq 3q \mid (\overline{\overline{G}}_{\beta,\gamma}\, \overline{\overline{D}})^{100\,q\cdot 2^n}\, \underbrace{(q,\dots,q)}_{m}\right) \geq 1 - q\cdot e^{-q}. \tag{19}$$

Before proving (17) and (19), let us explain why they imply (12). Let us consider probabilities of the event $\min_i(a_i) \geq 3q$ according to both measures in (17), where we choose $t = 100\,q\cdot 2^n$. Since the indicator function of this event is monotonic, (17) implies inequality of the same sense between these probabilities. But from (19) the right probability is not less than $1 - q\cdot e^{-q}$, so the left probability also is, which amounts to (12).

Now let us prove (17). For $t = 0$ this is evident. Then we argue by induction, but first prove that

$$Q\,(\overline{G}_{\varepsilon,\,0.1}\,\overline{D})^n \succ \overline{\overline{G}}_{\beta,\gamma}\,\overline{\overline{D}}\,Q, \tag{20}$$

where β and γ are defined in (18). It is evident that $Q\,\overline{D}^n \succ \overline{\overline{D}}\,Q$, whence $\overline{\overline{G}}_{\beta,\delta}\,Q\,\overline{D}^n \succ \overline{\overline{G}}_{\beta,\delta}\,\overline{\overline{D}}\,Q$. Let us prove that

$$Q(\overline{G}_{\varepsilon,\,0.1}\,\overline{D})^n \succ \overline{\overline{G}}_{\beta,\delta}\,Q\,\overline{D}^n. \tag{21}$$

To prove (21), it is sufficient to apply both sides to a measure concentrated in one configuration $a \in \mathbb{R}^q$ and couple them as follows: $\overline{\overline{g}}(i)$ equals δ if the event

$$\forall\, j \in [ni,\ n(i+1)-1]\ \ \exists\, t \in [1,\ n]\ :\ \overline{g}_{j,\,t} = 1 \tag{22}$$

takes place and equals zero otherwise. Here $\overline{\overline{g}}_i$ serve $\overline{\overline{G}}_{\beta,\delta}$ in the sense of (16) and $\overline{g}_{j,\,t}$ serve the t-th application of $\overline{G}_{\varepsilon,\,0.1}$ in the sense of (10). We assume that $\overline{g}_j$ are distributed as described in the definition of $\overline{G}_{\varepsilon,\,0.1}$. The probability of event (22) is $\{1-(1-\varepsilon)^n\}^n = \beta$, whence $\overline{\overline{g}}_i$ are distributed as declared in the definition of $\overline{\overline{G}}_{\beta,\delta}$. If $\overline{g}_{j,\,t} \equiv 0$, (21) is evident. Now let us see how the components of both sides increase if some $\overline{g}_{j,\,t} > 0$. For any $i,\ j = 0,\dots,q-1$ and any $k = 0, 1, 2, \dots$ we denote *Impact* $(i,\ j,\ k)$ the infimum of the fraction y/x, where y is the amount by which the i-th component of $\overline{D}^k\,a$ increases and x is the amount by which a_j increases, all the other components of a remaining unchanged. It is easy to prove by induction over k that

$$\forall\ i,j,k\ :\ \textit{Impact}\ (i,j,k) \geq \begin{cases} (1/2)^k & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the event (22) guarantees that the j-th component of $(\overline{G}_{\varepsilon,\,0.1}\,\overline{D})^n$ is greater than the j-th component of $\overline{D}^n\,a$ at least by $0.1\cdot(1/2)^n$ for all j in the range $ni \leq j < n(i+1)$, whence the i-th component of $Q\,(\overline{G}_{\varepsilon,\,0.1}\,\overline{D})^n\,a$ is greater than the i-th component of $Q\,\overline{D}^n\,a$ at least by $0.1\cdot(1/2)^n$, whence (21) follows. Thus (20) is proved, using which we can prove (17) by induction.

It remains to prove (19). For any $t\geq 0$ we can represent the measure $(\overline{\overline{G}}_{\beta,\gamma}\ \overline{\overline{D}})^t\ (q,\dots,q)$ as induced by i.i.d random variables $g_{i,t}$, which equal γ with probability β and 0 with probability $1-\beta$, with the initial condition $a_{i,0}\equiv q$ and inductive rule

$$a_{i,t} \;=\; \min(a_{i-1,t-1},\ a_{i,t-1},\ a_{i+1,t-1})+g_{i,t}-\frac{4}{m}\ \text{for}\ t>0,$$

where $i-1$ and $i+1$ are modulo m. Let us call level t the set $\{(i,t)\,:\,i=0,\dots,m-1\}$ and a path leading to a point (i,t) a sequence $s_1,s_2,\dots,s_t=i\in\{1,\dots,m-1\}$ such that $(s_k-s_{k+1})\in\{-1,0,1\}$ (modulo m) for all $k=1,\dots,t-1$. Let us call gain of this path the sum $g_{s_1,1}+\dots+g_{s_t,t}-4t/m$. It is evident that $a_{i,t}$ equals q plus the minimum of gains of all the paths leading to the point (i,t). (Essentially we are dealing here with a special case of first-passage percolation.) Therefore the inequality $a_{i,t}<3q$ is equivalent to existence of a path leading to (i,t) whose gain is less than $2q$. If gain of $s_1,\dots,s_t$ is less than $2q$, then

$$g_{s_1,1}+\dots+g_{s_t,t}\ \leq\ 2q+\frac{4t}{m}.$$

Therefore for any path leading to the level t, the probability that its gain is less than $2q$ does not exceed

$$\sum_{k=0}^{[h]}\binom{t}{k}\cdot\beta^k\cdot(1-\beta)^{t-k},\ \text{where}\ h\ =\ 10\cdot 2^n\cdot\left(2q+\frac{4t}{m}\right).$$

The number of paths that lead to the level t is $m\cdot 3^{t-1}$. Therefore

$$\text{Prob}\ (\min_i(a_{i,t})<3q)\ \leq\ m\cdot 3^{t-1}\cdot\sum_{k=0}^{[h]}\binom{t}{k}\cdot\beta^k\cdot(1-\beta)^{t-k}.$$

For any $x\geq 1$ this is less than

$$m\cdot 3^t\cdot\sum_{k=0}^{t}\binom{t}{k}\cdot x^{h-k}\cdot\beta^k\cdot(1-\beta)^{t-k}\ =\ m\cdot 3^t\cdot x^h\cdot\left(\frac{\beta}{x}+(1-\beta)\right)^t.\quad (23)$$

Let us take $x=100$ and remember that $\beta\geq 0.99$, $t=100\,q\cdot 2^n$ and $q\geq 1000\,n\cdot 2^n$, whence

$$h\ =\ 10\cdot 2^n\left(2q+\frac{4t}{m}\right)\ \leq\ 10\cdot 2^n(2q+40n\cdot 2^n)<30q\cdot 2^n.$$

Considering all this, (23) is less than $q\cdot e^{-q}$. Thus we have proved (19) which completes the proof of our main result, Theorem 2.

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