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Locally Interacting Systems and Their Application in Biology

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FOREWORD

A new branch of mathematics – the theory of systems with a large number of locally interacting random components – has developed in recent years. This theory is a natural instrument for the mathematical modeling of actually existing systems of various natures, such as complex biological, chemical, physical, and cybernetical systems and socioeconomic structures. The overlying problems involve many sciences, their urgency emphasized by the great interest shown by scientists of diverse schools and trends in many countries. Many ideas connected with Markov fields and interacting processes sprang up simultaneously and independently out of physical (Ising model) and biological (neuron-like models) problems. Later they were successfully applied to the description of other systems and, moreover, led to the creation of a new area of probability theory.

Realizing the importance of making this new development widely known and understood and the necessity for a wider scope of biologically oriented mathematical research, the Organizing Committee (R.L. Dobrushin, director) decided to hold a School-Seminar on this matter in Pushchino, Moscow Reg., USSR. It was organized by the USSR Academy of Sciences Research Computing Centre in cooperation with the Academic Institute of Information Transmission Problems. Ten 2-hour lectures and thirteen 1-hour communications were read at the School-Seminar; among these were:

E.B. Dynkin. On Gibbs representation of random fields (4 hours)
R.L. Dobrushin. Grounds for statistical mechanics: infinte-dimensional point of view (4 hours)
A.L. Toom. Random combines with local interaction (4 hours)
A.B. Katok. Entropy, variation principles and Gibbs measures for dynamic systems (6 hours)
R.A. Minlos. Generalized random processes (2 hours)

This collection, based on the communications delivered at the School-Seminar, has subsequently been re-edited and the results updated. Not all the articles in this collection are biologically oriented, but each serious investigation of a system with local interactions is important to biology. The collection is divided into two parts according to content, each part containing articles of topical similarity.

Part I includes articles that investigate systems whose specific characteristics are found in their deterministic part. Random noise is then added to the deterministic part, the only requirement being its conformity to certain weak conditions, such as its sufficiently small magnitude.

Systems with continuous space and time, including those affected by small random noise, are dealt with in Toom's article. Cirel'son's work had unexpected results: The author constructs a nonergodic one-dimensional random (true, non-uniform) system with local non-degenerate interaction. Shnirman's article contains new results on nonergodic systems. The main result of Kurdyumov's work is proof of algorithmic unsolvability of the problem of discriminating ergodic and nonergodic random media. He also investigates the possibilities of identifying formal languages by similar networks. Galperin's work is devoted to the behavior of deterministic monotone homogeneous systems. The work of Borisyuk et al. is the most biologically oriented of all.

Part II concerns works in which the probabilistic aspects of the theory are more explicitly revealed. The work by Averintsev, along with the review of results on Gibbs representation of random fields, specifies a class of fields described only by the pair potential. The articles by Stavskaya, Vasilyev, and Kryukov deal with various cases of an evoluting system having a stationary Gibbs distribution without assuming its reversibility in time. Stavskaya gives a criterion for the existence of distributions of a certain form. Kryukov's article investigates a system modeling certain types of neuron activity. Vasilyev's work contains, among other things, an example of a non-ergodic two-dimensional system built with the help of Ising's model. Chetayev develops a theory of multicomponent monotone systems. Molchanov's work states that the number of fields with a given Gibbs potential is substantially larger than the number of states at one point. This result is based on a peculiar nonlinear eigenvalue problem reported by E.B. Dynkin at the School-Seminar. Malyshev develops a spectral theory of interacting Markov processes on the spaces of states which are encountered in the quantum field theory. In Evstigneev's work, the ideas of discrete random fields are extended to continuous fields by means of a new interpretation of Gibbs representation.

Hopefully the present collection will draw the attention of mathematicians, biophycisists and engineers to this promising development.

Editors

MONOTONIC EVOLUTIONS IN REAL SPACES

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I. Introduction

This paper succeeds the works |I,2,3| in investigating uniform multi-component developing systems with local interactions. However, in this paper the "components" are disposed not in a lattice Z^{d+I}, as in the cited works, but in a real space R^{d+I}, whence they should not be termed automata. On the other hand, every component dealt with here has two states only, and a state of the whole system can be completely determined by a subset of components. Thus, we shall speak no more of states of components, but only of subsets of R^{d+I}. Let us give some definitions.

Throughout this paper, V is a half of a (d+I)-dimensional real space: $V = R^d \cdot R_+$. Elements of R^{d+I} are called points and denoted by (s,t), $s \in R^d$, $t \in R$. So $V = \{(s,t): t \geq 0\}$. Subsets of V (perhaps, improper) are called states. A number $r \geq I$ and sets

$$U = \{(s,t) : |s| \leq r, \quad -r \leq t \leq -I\} , \qquad I = \{(s,t) : 0 \leq t < r\}$$

are given. Subsets of I are called initial states, or shorter bases. Signs +,-, when applied to points and point sets, denote vector addition and substraction. For any $A \subset R^{d+I}$, $B \subset R^{d+I}$, $C \in R^{d+I}$, $k \in R$:

$A+c=\{a+c,\ a\in A\}$, $A+B=\{a+b,\ a\in A,\ b\in B\}$, $kA=\{ka,\ a\in A\}$, $-A=(-I)\cdot A$.

An ensemble E is given, the elements of which are among the subsets of U. A state $X \subset V$ is called a trajectory, if

(I) $$\forall v \in V \setminus I : v \in X \Longleftrightarrow (U \cap (X - v)) \in E.$$

LEMMA I. For any base $B \subset I$ there is just one trajectory X, such that $B = I \cap X$.

Proof. We cut the set $V \setminus I$ into strata with a thickness of I:

$$V_k = \{(s,t) : r+k-I \leq t < r+k\} , \quad k=I,2,3,\ldots$$

Of course, any state X is determined by its intersections $X \cap I$, $X \cap V_I$,

$X \cap V_2, \ldots$ On the other hand,

(2) $$v \in V_k \Rightarrow v+U \subset (I \cup V_1 \cup \ldots \cup V_{k-1}).$$

First, let us prove that there is a trajectory. We take $X \cap I=B$ and define $X \cap V_k$ inductively. A point $v \in V_k$ enters $X \in V_k$, if

$$((X \cap (I \cup V_1 \cup \ldots \cup V_{k-1}))-v) \cap U \in E.$$

Evidently, this definition is consistent and X so defined is a trajectory.

Now we suppose that X and Y are different trajectories and $X \cap I=Y \cap I$. We choose the least k for which $X \cap V_k \neq Y \cap V_k$. We may assume that $v \in X \cap V_k$, but $v \notin Y \cap V_k$. This implies, that $(U \cap (X-v)) \in E$, but $(U \cap (Y-v)) \notin E$, which contradicts (2) and proves the lemma.

We call the whole described construction an <u>evolution</u>. So, evolution $\mathcal{E}$ is determined by dimension d, radius r, and ensemble E. From LEMMA I, we denote the trajectory by $T_B(\mathcal{E})$, for which $T_B \cap I=B$.

We call an evolution <u>monotonic</u> if

(3) $$A \in E,\ A \subset D \subset U \Rightarrow D \in E.$$

There is a natural duality among evolutions with fixed d and r. For an evolution $\mathcal{E}$, the <u>dual</u> evolution, evolution $\bar{\mathcal{E}}$ is defined by the formula

(4) $$\forall A \subset U: (A \in \bar{E} \Leftrightarrow (U \setminus A) \notin E).$$

Of course, $\bar{\mathcal{E}}$ is monotonic, if so is $\mathcal{E}$. Trajectories of $\bar{\mathcal{E}}$ are supplements for trajectories of $\mathcal{E}$ in V.

<u>This paper deals with monotonic evolutions only,</u> and "evolution" means "monotonic evolution" everywhere below. In fact, we assume also that

(5) $$\emptyset \notin E,\ U \in E.$$

This causes no loss of generality, because otherwise E would contain either none or all the subsets of U, which would be trivial. The condition (5) implies that the empty subset of V and V itself are both trajectories. These are analogous because of the duality, and we shall confine ourselves to the empty trajectory and its perturbations.

We shall write $\mathcal{E}_1 < \mathcal{E}_2$, if evolutions $\mathcal{E}_1$ and $\mathcal{E}_2$ have the same d and r, and $E_1 \subset E_2$.

LEMMA 2. $\mathcal{E}_I < \mathcal{E}_2$ infers $T_B(\mathcal{E}_I) \subset T_B(\mathcal{E}_2)$ for any B base. This can be proved by induction, as for LEMMA I.

2. Destroying Evolutions

DEFINITION I. We call the value (perhaps, infinite) of sup t for $(s,t) \in T_B$ the lifetime of a base B. We call an evolution destroying if any limited base has a finite lifetime in it. We call an evolution linearly destroying if it has such a λ that any base fitting into a ball with a radius of ρ has a lifetime no longer than $\lambda\rho$.

This section shows (not completely) which evolutions are destroying and which of these are linearly destroying. For that it is convenient to determine evolutions not with their ensembles E, but with some other ensembles; we are going to introduce them.

DEFINITION 2. We call a set $C \subset U$ an obstacle if $C \cap A = \emptyset \Rightarrow A \notin E$. An ensemble H of obstacles is called a handicap if all obstacles have an element of H as a subset.

LEMMA 3. Let H be a hadicap of an $\mathcal{E}$ evolution. A subset of U enters the ensemble E if, and only if it meets all the elements of H.

Proof. Let $A \in E$. Hence, directly from DEFINITION 2, A meets all obstacles including all the elements of H. Now, let $A \subset U$, and $A \notin E$. Suppose A meets all elements of H. So A meets all obstacles. Therefore, $U \setminus A$ is not an obstacle; that is, we have a $C \in E$, so that $C \cap (U \setminus A) = \emptyset$. Hence, $C \subset A$. But, combined with $C \in E$ and $A \notin E$, this contradicts the monotony of $\mathcal{E}$, which proves the lemma.

This lemma shows that though handicaps of an evolution may differ, each of them determines a unique evolution. Using this, we shall further determine an evolution by its handicap H, the number r not being mentioned. Instead of the condition (I), defining a trajectory, we shall use its equivalent:

$$v \in X \iff (\forall A \in H : (X-v) \cap A \neq \emptyset). \tag{6}$$

We can also say that evolutions $\mathcal{E}_I$ and $\mathcal{E}_2$ are one and the same if they have a common handicap. Correspondingly, $\mathcal{E}_I < \mathcal{E}_2$ if any obstacle of $\mathcal{E}_2$ is an obstacle of $\mathcal{E}_I$.

LEMMA 2 thus also remains true.

We shall use the following denotations: $\mathrm{ray}(A) = \bigcup_{k\geq 0} kA$; $\mathrm{clos}(A)$ is the closure of A; $\mathrm{conv}(A)$ is the convex hull of A for any A set in R^{d+I}. The following set is of importance here:

$$\sigma = \bigcap_{A\in H} \mathrm{ray}(\mathrm{clos}(\mathrm{conv}(A))).$$

It is clear that σ is the same for any H handicap of a given evolution $\mathcal{E}$. So we may say $\sigma = \sigma(\mathcal{E})$.

THEOREM I. An $\mathcal{E}$ evolution is linearly destroying if and only if $\sigma(\mathcal{E}) = \{O\}$; that is, $\sigma(\mathcal{E})$ consists of the only O origin of coordinates.

The following lemma is used not only to prove this theorem.

LEMMA 4. The condition $\sigma(\mathcal{E}) = \{O\}$ is equivalent to the following. There are n homogeneous linear functions $L_I,\ldots,L_n : R^{d+I} \to R$, where $n \leq d+2$, and a number $k>0$, so that: (I) $\sum_{\nu=I}^{n} L_\nu(s,t) = kt$ for any (s,t); (2) the set $\{v \in U : L_\nu(v) \geq 0\}$ is an obstacle for any ν from I to n.

Proof. First assume that there are $L_I(.),\ldots,L_n(.)$ and $k>0$ with the properties (I),(2) and prove that $\sigma = \{O\}$. Assume the contrary: $v = (s,t) \in \sigma$, $v \neq O$. Of course, $t<0$ here. Since $\sum_{\nu=I}^{n} L_\nu(v) = kt \leq 0$, we can choose ν which makes $L_\nu(v)<0$, whence $L_\nu(lv)<0$ for all $l>0$. Therefore, any $\mathrm{clos}(\mathrm{conv}(A))$, and accordingly any $A \in H$ contains an a point in which $L_\nu(a)<0$. But this contradicts the fact that the set $\{a \in U : L_\nu(a) \geq 0\}$ is an obstacle.

Now we assume $\sigma = \{O\}$ and seek $L_I(.),\ldots,L_n(.)$, $k>0$. Let F_A, where $A \in H$, stand for the set of homogeneous linear normed functions $f: R^{d+I} \to R$, so that $f(v) \leq 0$ for all $v \in \mathrm{ray}(\mathrm{clos}(\mathrm{conv}(A)))$. Denote the union of these F_A for all $A \in H$ by F. Of course,

$$\mathrm{ray}(\mathrm{clos}(\mathrm{conv}(A))) = \{v : \forall f \in F_A\ (f(v) \leq 0)\}$$

for any $A \in H$. Therefore

$$\{v : \forall f \in F\ (f(V) \leq 0)\} = \{O\}.$$

We introduce also a set $C = \{(s,t) : t \leq -I\}$ and apply the theorem 2I.3 in |4| (a generalized variant of Helly's theorem) to the F family and the C set. This theorem says that there are n functions $f_I,\ldots,f_n \in F$, where $n \leq d+2$, and positive numbers $\lambda_I,\ldots,\lambda_n,\varepsilon$, so that $v \in C$: $\sum_{\nu=I}^{n} \lambda_\nu f_\nu(v) \geq \varepsilon$. Therefore $\sum_{\nu=I}^{n} \lambda_\nu f_\nu(s,t) = -kt$, where $k>0$. The functions $L_\nu = -\lambda_\nu f_\nu$, $I \leq \nu \leq n$, and the number k are the ones we are seeking. The lemma is proved.

<u>Proof</u> of theorem I. First assume, that $\sigma = \{O\}$. There are $L_I(.),\ldots,$ $L_n(.)$, $k > O$, possessing the properties, resulting from LEMMA 4. Using the property (2), we can prove by induction, that

$$\sup_{v \in T_B} L_\nu(v) \leq \sup_{v \in B} L_\nu(v)$$

for any B base and any ν from I to n. This and the property (I) infer

$$(s,t) \in T_B \Rightarrow kt = \sum_{\nu=I}^{n} L_\nu(s,t) \leq \sum_{\nu=I}^{n} \sup_{v \in B} L_\nu(v).$$

The last term does not exceed const. ρ, where ρ is the radius of a ball containing B. This is evident if the ball's center lies on axis t. So this is correct in general, because the last term's value is invariant for a translation of B by any vector (s,O).

Now assume that $\sigma \neq \{O\}$. So σ meets the hyperlane $t = -I$, say in point $(s^O,-I)$. Let us pass on to oblique coordinates $s' = s+ts^O$, $t'=t$. Accordingly, our evolution $\mathcal{E}$ transforms into another evolution $\mathcal{E}'$. Of course, $\mathcal{E}'$ is linearly destroying if so is $\mathcal{E}$. The set $\sigma(\mathcal{E}')$ contains the ray $\{(O,t) : t \leq O\}$. Now we introduce a set Q by the formula

$$Q = \{(s,t) \in V : |s|^2 < \rho^2 - tR^2\} .$$

This part of the theorem is inferred by the fact that

$$B = Q \cap I \Rightarrow Q \subset T_B,$$

which we shall prove by induction. We have to prove for any point $(s,t) \in Q \setminus I$, that

$$(7) \qquad (Q \cap ((s,t) + U)) \subset (T_B \cap ((s,t) + U)) \Rightarrow (s,t) \in T_B.$$

This is evident if s=O, so now let $s \neq O$. Let us introduce a vector $s' \in R^d$, such that (I) $s'=ks$, where $k > I$; (2) $|s'|^2 < \rho^2 - tR^2$. Let π be the hyperplane passing the point $(s';t)$ and orthogonal to the vector (s,O). Let η be the half-space, bounded by π, which contains the origin of the coordinates. It can be deduced geometrically that

$$(\eta \cap ((s,t) + U)) \subset (Q \cap ((s,t) + U)).$$

This and the left side term in (7) infer

$$(\eta \cap ((s,t) + U)) \subset (T_B \cap ((s,t) + U)).$$

Therefore, for any A obstacle, the set $(s,t) + A$ meets T_B, whence

$(s,t) \in T_B$ by the criterion (6). Theorem I is proved.

LEMMA 5. Let a Q set in R^{d+I} have the following property:

(8) $$v \in Q \cap (V \setminus I) \Rightarrow (v+A) \cap Q \neq \emptyset$$

for any A obstacle. Then, taking $B = Q \cap I$, we have $Q \subset T_B$.

Proof is by induction.

THEOREM 2. If $\sigma(\mathcal{E}) \neq \{0\}$, then every one of the following three assumptions:

(I) the d dimension equals I;
(2) the $\mathcal{E}$ evolution has a finite handicap, and
(3) the set $\sigma(\mathcal{E})$ is (d+I)-dimensional
infers that the $\mathcal{E}$ evolution is not destroying. Moreover, each of the assumptions (I) and (2) provides a B limited base, for which $-\sigma(\mathcal{E}) \subset T_B$.

Proof in the cases (I) and (2) boils down to finding a non-empty limited set P, such that the set $Q = P-\sigma$ conforms to the formula (8).

Case I. If d=I, we may take an open circle with a radius of 2r and a center of (0,-2r) for P. In this case the proof of (8) is evident.

Case 2 calls for a definition and two lemmas.

DEFINITION 3. We call a set $P \subset R^{d+I}$ obtuse for a set $A \subset R^{d+I}$, if $\forall v \in R^{d+I}$: $((P+v) \cap A = \emptyset \Rightarrow (P+v) \cap \mathrm{conv}(A) = \emptyset)$.

LEMMA 6. If P is obtuse for A, then any P+P' is obtuse for A.

Proof is evident.

LEMMA 7. The set $-(d+I)\mathrm{conv}(A)$ is obtuse for any A.

Proof. First let A consist of d+2 points not belonging to a hyperplane. Let $P=w-(d+I)\mathrm{conv}(A)$ and $P \cap A = \emptyset$. Let $\eta: R^{d+I} \to R^{d+I}$ be such a homothetic transformation that $\eta(\mathrm{conv}(A)) = P$. Of course, the coefficient of η equals $-(d+I)$. We denote the center of η by ξ, the elements of A by a_k, and $v_k=\eta(a,k)$, where $I \le k \le d+I$. Let $L_k : R^{d+I} \to R$, $I \le k \le d+2$, be such a linear function that $L_k(v_k)=I$, $L_k(v_l)=0$ for all $l \neq k$. It is evident that $\sum_{k=I}^{d+2} L_k(v)=I$ for any $v \in R^{d+I}$, that

$$P = \{v : \forall k \; (L_k(v) \geq 0)\},$$

and that $\sum_{k=I}^{d+2} L_k(v_k) = d+2$. The latter, combined with

$$L_k(v_k) - L_k(\xi) = (d+I)(L_k(\xi) - L_k(a_k)),$$

infers that $\sum_{k=I}^{d+2} L_k(a_k) = 0$. On the other hand, for any k, as $a_k \in P$, there is such an $l(k)$ that $L_{l(k)}(a_k) < 0$. Here $l(k)$ cannot equal k in all cases. So we have $k, l, k \neq l$, such that $L_l(a_k) < 0$. Therefore max $L_l(v) < 0$ when v runs all A and all conv(A) as well. The latter infers $P \cap \text{conv}(A) = \emptyset$, q.e.d.

This obviously infers the lemma for a case when A consists of all vertices of any simplex in R^{d+I}.

Now let A be any set. Let $P = w - (d+I)\text{conv}(A)$ and P meet conv(A). Let $\eta : R^{d+I} \to R^{d+I}$ be the same as above. Since $P = \eta(\text{conv}(A))$ meets conv(A), the center ξ belongs to conv(A). From Caratheodory's theorem, ξ belongs to a simplex S, of which the vertices belong to A. Of course, S meets $\eta(S)$, whence, by the proof above, $\eta(S)$ contains a vertex of S. So, P meets A, q.e.d.

Now we can examine Case 2. We take

$$P = \text{int}\left(-D \sum_{A \in H} \text{conv}(A) + Sp(I) + w\right).$$

Here int(.) denotes the set of inner points. D is more than d+I. H is a finite handicap. Sp(I) denotes the ball with a center of O and a radius of I; it is added to guarantee that P is non-empty. A vector w is added to make $\forall (s,t) \in P : t \leq 0$. It is easy to prove that P is obtuse for every $A \in H$. Therefore $Q = P - \sigma$ also is obtuse for every $A \in H$. Let us prove that Q conforms with the formula (8). Let $v \in Q \cap (V \setminus I)$. So $v = (s^I, t^I) + (s^2, t^2)$, where $(s^I, t^I) \in P$, $(s^2, t^2) \in -\sigma$, $t^I + t^2 \geq r$. Here $t^I \leq 0$, whence $t^2 \geq r$. Therefore, for any $A \in H$, the set $(s^2, t^2) + \text{clos}(\text{conv}(A))$ meets $-\sigma$, whence $v + \text{clos}(\text{conv}(A))$ meets Q. Since Q is open, $v + \text{conv}(A)$ also meets Q, and since Q is obtuse for A, $v + A$ also meets Q, q.e.d.

<u>Case 3</u>. σ's being (d+I)-dimensional infers that there is an l straight line passing through the O origin, and that there is an $\varepsilon > 0$, for which any parallel of l, whose distance from l is less than ε, meets conv(A) for any A obstacle. Making use of a coordinate transformation, as in THEOREM I, we can assume that l is the axis t. We take $Q = l + P$, where P is a ball with a radius of $R^2 \varepsilon^{-I}$. It is easy to prove that (8) holds for this Q.

THEOREM 3. <u>For any $\mathcal{E}$ evolution and any B limited base there is such a number ϕ that</u> $T_B(\mathcal{E}) \subset Sp(\phi) - \sigma(\mathcal{E})$.

Its proof requires two lemmas.

<u>LEMMA IO</u>. For any A obstacle and any B base

$$T_B \subset B - \text{ray}(\text{conv}(A)).$$

Proof. First let us prove that

$$(9) \qquad T_B \subset B \cup (B-A) \cup (B-A-A) \cup (B-A-A-A) \cup \ldots = J.$$

The latter equality defines J. Let (9) be proved for the strata $V_I, \ldots V_{k-I}$ introduced in LEMMA I. Let $v \in V_k \cap T_B$. By criterion (6), v+A meets T_B. As v+A lie in the previous strata, v+A also meets J, whence $v \in J$. So the formula (9) is correct. It remains correct when substituting conv(A) for A. But C+C=2C for any C convex. Therefore,

$$T_B \subset \bigcup_{k \in Z_+} (B - k\ \mathrm{conv}(A)) \subset \bigcup_{k \geq 0} (B - \mathrm{conv}(A)) = B - \mathrm{ray}(\mathrm{conv}(A)),$$

q.e.d.

LEMMA II. Let F be a non-empty ensemble of non-empty closed sets in R^{d+I}. Then for any $\rho > 0$ there is such a $\phi > 0$ that

$$\bigcap_{A \in F} (\mathrm{ray}(A) + Sp(\rho)) \subset \bigcap_{A \in F} \mathrm{ray}(A) + Sp(\phi).$$

Proof. The contrary implies that there is such a $\rho > 0$ that

$$\forall\ \phi > 0 : \bigcap_{A \in F} (\mathrm{ray}(A) + Sp(\rho)) \setminus (\bigcap_{A \in F} \mathrm{ray}(A) + Sp(\phi)) \neq \emptyset.$$

Let us apply a homothetic transformation with a center of O and a coefficient of ϕ^{-I}, and denote $\varepsilon = \rho\phi^{-I}$. We obtain

$$\forall\ \varepsilon > 0 : (\bigcap_{A \in F} (\mathrm{ray}(A) + Sp(\varepsilon)) \setminus (\bigcap_{A \in F} \mathrm{ray}(A) + Sp(I)) \neq \emptyset.$$

This and compactness demand that

$$\bigcap_{\varepsilon > 0} \bigcap_{A \in F} (\mathrm{ray}(A) + Sp(\varepsilon)) \setminus (\bigcap_{A \in F} \mathrm{ray}(A) + \mathrm{int}(Sp(I))) \neq \emptyset,$$

whence

$$\bigcap_{A \in F} \mathrm{ray}(A) \setminus (\bigcap_{A \in F} \mathrm{ray}(A) + \mathrm{int}(Sp(I))) \neq \emptyset,$$

which is obviously wrong.

Proof of THEOREM 3. LEMMA IO infers

$$T_B \subset \bigcap_{A \in H} (Sp(\rho) - \mathrm{ray}(\mathrm{clos}(\mathrm{conv}(A)))),$$

where $Sp(\rho)$ contains B. Now LEMMA II infers the theorem.

3. Evolutions with Random Noises

Here evolutions with small random noises are examined. This paper deals with a very specific noise, but we shall generalize our results later. Let $\delta > 0$. We denote by a δ- brick any set of the following form:

$$\{(s,t) : p_k\delta \le s_k \le (p_k+I)\delta,\ I\le k\le d,\quad p_O\delta\le t\le(p_O+I)\delta\},$$

where p_O, $p_I,\dots,p_d$ are integers. We denote by $V(\delta)$ the set of δ-bricks. We shall use the following denotations for any countable W set: $\Omega_W = \{O;I\}^W$. Elements of Ω_W are $\omega = (\omega_a)$, $\omega_a \in \{O;I\}$, $a \in W$. Further, μ_W is the set of normed measures on Ω_W (in fact, on the σ-algebra, generated by cylinder sets in Ω_W). For any ε, $O\le\varepsilon\le I$, we introduce a subset $\mu_W(\varepsilon) \subset \mu_W$. A measure $\mu \in \mu_W$ enters $\mu_W(\varepsilon)$, if

$$\mu(\omega_a = I \text{ for all } a \in A) \le \varepsilon^{|A|}$$

for any finite $A \subset W$, where $|.|$ denotes cardinality. Of course, $\varepsilon_I<\varepsilon_2 \Rightarrow$ $\Rightarrow \mu_W(\varepsilon_I) \subset \mu_W(\varepsilon_2)$. Let us choose an evolution $\mathcal{E}$. For any $\delta > O$ we introduce a map F_δ acting on $\Omega_{V(\delta)}$. For any $\omega \in \Omega_{V(\delta)}$ the image $F_\delta(\omega)$ is a state in V defined as follows. A point $v \in V$ enters $F_\delta(\omega)$ if at least one of the following two conditions holds: (I) $\omega_x = I$, where x is a δ-brick containing v; and (2) $v \in V\setminus I$ and $(U \cap (F_\delta(\omega) - v)) \in E$. As for LEMMA I, it is easy to prove by induction that there is a unique $F_\delta(\omega)$ for any $\omega \in \Omega_{V(\delta)}$. Let us denote by $P_\delta(v,\mu)$ the probability in a measure μ on $\Omega_{V(\delta)}$, that $v \in F_\delta(\omega)$. We also denote

$$P_\delta = \lim_{\varepsilon\to O} \sup_{\substack{v\in V \\ \mu\in\mu_{V(\delta)}(\varepsilon)}} P_\delta(v,\mu).$$

The limit exists, because the supremum depends on ε monotonically.

<u>LEMMA I2.</u> The value of P_δ does not depend on $\delta > O$.

Its proof is based on the following lemma.

<u>LEMMA I3</u>. V_I and V_2 are countable sets. To every $x \in V_I$ there corresponds a set $N(x) \subset V_2$, and there are C_I, C_2 constants, such that

(IO) $\forall x \in V_I : I \le |N(x)| \le C_I;\qquad \forall y \in V_2 : I \le |\{x : y \in N(x)\}| \le C_2.$

A map $G: \Omega_{V_2} \to \Omega_{V_I}$ is defined by the equality:

$\omega_x = \max_{y\in N(x)} \omega_y$ for all $x \in V_I$. Then for any $\varepsilon_I > O$ there is such $\varepsilon_2 > O$, that for any $\mu \in \mu_{V_2}(\varepsilon_2)$ the measure on Ω_{V_I}, induced by μ with the map G, belongs to $\mu_{V_I}(\varepsilon_I)$.

<u>Proof</u>. We leave it to the reader to examine two particular cases, when $C_I=I$ and when $C_2=I$. If $C_I=I$, one may take $\varepsilon_2 = (\varepsilon_I)^{C_2}$. If $C_2=I$,

one may take $\varepsilon_2 = I-(I-\varepsilon_I)^{I/C_I}$. The lemma can be reduced to these cases by introducing an intermediate set

$$V_3 = \{(x,y) : x \in V_I,\ y \in N(x)\}$$

and maps $G_I : \Omega_3 \to \Omega_I$ and $G_2: \Omega_2 \to \Omega_3$ defined by the following formulae:

$$\omega_x = \max_{y \in N(x)} \omega_{(x,y)}, \quad \omega_{(x,y)} = \omega_y, \quad x \in V_I, \quad y \in N(x).$$

Of course, $G = G_I G_2$. The maps G_I and G_2 conform to the particular cases examined. Therefore, in the general case, one may say ε_2 depends on ε_I by a superposition of the dependencies mentioned above:

$$\varepsilon_2 = (I - (I-\varepsilon_I)^{I/C_I})^{C_2}$$

<u>Proof</u> of LEMMA I2. Let δ_I and δ_2 be values of δ. We shall prove that $P_{\delta_I} \geq P_{\delta_2}$ by assotiating such an $\varepsilon_2 > 0$ with any $\varepsilon_I > 0$ that

(II) $\quad \forall v \in V: \quad \sup_{\mu \in \mu_{V(\delta_I)}(\varepsilon_I)} P_{\delta_I}(v,\mu) \geq \sup_{\mu \in \mu_{V(\delta_2)}(\varepsilon_2)} P_{\delta_2}(v,\mu).$

For any δ_I -brick x we denote by N(x) the set of δ_2 - bricks meeting x. Of course, (IO) holds. The dependence of ε_2 on ε_I, resulting from LEMMA I3 for this case, infers (8). In fact, the left side of (8) is not less, and its right side is not more than supremum of $Q_{\delta_2}(v,\mu)$ over all measures $\mu \in \mu_{V(\delta_2)}(\varepsilon_2)$. Here the value of $Q_{\delta_2}(v,\mu)^2$ is the probability, in a measure μ on Ω_{δ_2}, of the fact that $v \in F_{\delta_I}(G(\omega))$, where G was introduced in LEMMA I3.

Basing ourselves on LEMMA I2, we omit the index δ and now write $P(\mathcal{E})$ for P_δ. In this section we shall prove the following theorems.

THEOREM 4. <u>If $\mathcal{E}$ evolution is linearly destroying</u>, $P(\mathcal{E})=0$.

THEOREM 5. <u>If $\mathcal{E}$ evolution is non-destroying</u>, $P(\mathcal{E})=I$.

An analogue of these theorem for a discrete case is published as Theorem 5 in |3|. But in the present case, unlike the discrete one, non-linearly destroying evolutions are possible (see EXAMPLE 2 below). We do not know the value of $P(\mathcal{E})$ in this case. Perhaps $P(\mathcal{E})=I$ for all of them. Now we shall present some lemmas on which proofs of the theorems are based. Let us say that an $\mathcal{E}$ evolution is δ-<u>brick</u> if it has a handicap all elements of which are unions of δ-bricks.

LEMMA I4. Any δ-brick evolution, $0<\delta<1$, transforms any base consisting of δ-bricks into a trajectory consisting of δ-bricks. For any δ-brick evolution, $0<\delta<1$, and any $\omega \in \Omega_{V(\delta)}$ the image $F_\delta(\omega)$ consists of δ-bricks.

Proof. The two assertions of the lemma are similar, so we shall prove only the former. We cut the set V into strata having a thickness of δ:

$$V_k = \{(s,t) : (k-1)\delta \le t \le k\delta\}, \quad k = 1,2,3,\dots$$

and prove by induction, that if a B base consists of δ-bricks, so does every set $V_k \cap T_B$. Let this be proved for $V_1,\dots,V_{k-1}$. Now we apply the formula which can be proved directly:

$$T_B \cap V_k = \Big(\bigcap_{A \in H} \Big(T_B \cap \Big(\bigcup_{l=1}^{k-1} V_l\Big)\Big) - A\Big) \cap V_k .$$

In general, if C and D sets consist of δ-bricks, $C \cap D$ and $C-D$ do too. This fact applied to the right part of our formula proves the induction step, q.e.d.

LEMMA I5. $\mathcal{E}_1 < \mathcal{E}_2$ infers $P(\mathcal{E}_1) \le P(\mathcal{E}_2)$.

Proof is easy, provided some considerations of monotony are involved, which are developed in |5| , §2, in particular.

LEMMA I6. For any $\mathcal{E}$ non-destroying evolution there is such a B limited base that

$$\forall t \ge 0 \quad \exists s : (s,t) \in T_B(\mathcal{E}).$$

Proof. There is such a B' limited base, that

$$\forall t \ge 0 \quad \exists s, t' \ge t : (s,t') \in T_{B'}(\mathcal{E}).$$

This implies, that

$$\forall t \ge 0 \quad \exists s,t' : \ t \le t' \le t+r, \ (s,t') \in T_{B'}(\mathcal{E}),$$

since, this being wrong for some $t \ge 0$, the trajectory $T_{B'}$ would contain no points $(s,t') : t' \ge t$. Let $Q = T_{B'} + \{(0,t) : -r \le t \le 0\}$. Of course,

$$\forall t \ge 0 \quad \exists s : (s,t) \in Q.$$

Hence, the base $B = Q \cap I$ is the one sought, since $Q \subset T_B$.

Proof of THEOREM 4. For δ-brick evolutions this is an evident corollary of Theorem 5 in |3| and our LEMMA I4. Now let $\mathcal{E}$ be any linearly destroying evolution. So $\sigma(\mathcal{E}) = \{0\}$ by THEOREM I. Let us cover every A obstacle of $\mathcal{E}$ by the union A_δ of δ-bricks meeting A and take the set of these A_δ as a handicap determining a new evolution $\mathcal{E}_\delta$. One can prove (with LEMMA 4 it is quite easy) that $\sigma(\mathcal{E}_\delta)=\{0\}$ for sufficiently small $\delta>0$. So $P(\mathcal{E}_\delta) = 0$ for these δ. On the other hand, $\mathcal{E} < \mathcal{E}_\delta$ for any δ. Therefore, $P(\mathcal{E}) = 0$, q.e.d.

Proof of THEOREM 5. Let $\delta= I$. In fact, we shall prove that for any $\varepsilon> 0$ there is such a $\mu \in \mu_{V(I)}(\varepsilon)$, that

$$\sup_{v\in V} P_I(v,\mu) = I.$$

For this μ we shall take a Bernoully measure μ_ε, in which

$$\mu_\varepsilon(\omega_x = I \text{ for all } x \in A) = \varepsilon^{|A|}$$

for all finite $A \subset V(I)$.

Based on LEMMA I6, we have such a B limited base that

$$\forall t \geq 0 \quad \exists s : (s,t) \in T_B.$$

We denote by s(t) the value of s associated with t in this formula; that is,

$$\forall t \geq 0 : (s(t),t) \in T_B.$$

The properties of the map F_I infer that for any $(s^I,t^I) \in V$ and $t\geq 0$

$$(B + (s^I,t^I)) \subset F_I(\omega) \Rightarrow (s^I+s(t),\ t^I+t) \in F_I(\omega).$$

Denoting $s^I+s(t) = s^O$, $t^I+t = t^O$, we reword this in the following form. The fact of $(s^O,t^O) \in F_I(\omega)$ is guaranted if there is such a t, $0\leq t\leq t^O$, that

$$B + (s^O-s(t),t^O-t) \subset F_I(\omega).$$

The latter condition is ensured if ω is such that: "$\omega_x = I$ for all δ-bricks x, which meet $B + (s^O-s(t),t^O-t)$". The probability of the quoted condition in μ_ε is not less than a positive constant $\gamma> 0$ for all t (provided δ,ε, B are fixed). On the other hand, for values of t being only multiples of r + I, the quoted conditions are independent of each other. Therefore, the probability of the fact that at least one of the quoted conditions holds, is not less than

$$I - (I-\gamma)^{t^O/r+I}$$

which tends to I, when $t^{O} \to \infty$, q.e.d.

4. Examples

Here we produce three particular evolutions. We shall determine each of them by means of H, one of its handicaps. In all the cases d = 2; that is, $V = R^2 \cdot R_+$. All $A \in H$ lie in the plane t = -I. So, as a matter of fact, the time is discrete, and we may consider only integer values of t.

EXAMPLE I. We choose in the plane t = -I an equilateral triangle T with a center (O,O,-I). The elements of H are polygonal domains in T, of which the areas equal half the area of T.

It is easy to check that this evolution $\mathcal{E}$ is linearly destroying. We may consider its dual evolution $\bar{\mathcal{E}}$. Evidently, $\bar{\mathcal{E}} < \mathcal{E}$, whence $\bar{\mathcal{E}}$ is also linearly destroying. Imagine $\mathcal{E}$ functioning in a two-way noise, which would randomly add or cut out δ-bricks. Both empty and "all-V" trajectories are stable in $\mathcal{E}$ in the presence of such noise.

EXAMPLE 2. Elements of H are halves, that is, arcs of $I8O^{O}$, of the circle

$$\{(s,t) : |s| = I,\ t = -I\} .$$

This evolution is destroying, but not linearly. If the base is a full circle with a radius of r_O in the plane t = O, the trajectory consists of circles with radiuses of $r_t = (r_O^2 - t)^{I/2}$ for t = O,I,2,... The lifetime of this base is $(r_O)^{I/2} + O(I)$. We do not know what would happen if random noise were added.

EXAMPLE 3. We choose in the plane t = -I a square Q with a side of I and a center of (O,O,-I). The elements of H are quadruples of points lying one-to-one in all the sides of Q.

This evolution is non-destroying. A square Q' in the plane t = O with a side of $2^{I/2}$ and sides parallel to diagonals of Q has an infinite lifetime, because it produces equal squares at times t = I,2,... But THEOREM 2 is not applicable here since none of its three assumptions holds.

References

I. Vasilyev N.B., Petrovskaya M.B., Paytetski-Shapiro I.I. Models of voting with random error, Avtomat. i Telemeh. IO (I969), IO3-IO7.
2. Toom A.L. Monotone binary Tessellation Automata, Problemy Peredači Informacii I2 (I976), vyp.I, 48-54.

3. Toom A.L. Stable and Attractive Trajectories in multicomponent systems, Mnogokomponentnie Sluchainie Systemy, Moscow, Nauka, I977.
4. Rockafellar R.T. Convex Analysis, Princeton, New Jersey, Princeton Univ. Press, I970.
5. Mityushin L.G. Nonergodicity of uniform threshold networks for small selfexcitation, Problemy Peredači Informacii 6 (I970), vyp. 3, 99-I03.