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Markov Processes and Related Fields

Pseudo-Pinning in a Growth Model*

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Abstract. Velocity of growth is estimated in a discrete-time random process with local interaction with $\mathbf{R}^{\mathbf{Z}}$ as the configuration space. For any $\alpha,\ \beta \in [0,1]$, we consider an operator $P = R_{1,\beta} D_\alpha$, which acts on the set of normed measures on $\mathbf{R}^{\mathbf{Z}}$, where $D_\alpha : \mathbf{R}^{\mathbf{Z}} \to \mathbf{R}^{\mathbf{Z}}$ is defined by

$$\forall\ s \in \mathbf{Z} : y(s) = \alpha x(s) + (1-\alpha) \min(x(s-1),\ x(s),\ x(s+1))$$

and $R_{1,\beta}$ independently increases every component by 1 with a probability β. The velocity of growth is defined as

$$\mathrm{Vel}(P) = \lim_{t\to\infty} \mathsf{E}(P^t\, \delta_0)/t$$

where $\mathsf{E}(\mu)$ is the expectation of a component and δ_0 is the initial measure concentrated in the configuration "all zeros".

Theorem. There are positive constants $C_1,\ C_2,\ C_3,\ C_4$ and β_0 such that

$$C_1 \cdot \alpha^{C_2 \ln(1/\beta)/\beta} \le \mathrm{Vel}(P) \le C_3 \cdot \alpha^{C_4/\beta}$$

for all $0 \le \alpha < 1/2$ and $0 < \beta < \beta_0$.

The fact that the velocity tends to zero so fast when $\alpha,\ \beta \to 0$ is called pseudo-pinning by an analogy with the well-known pinning, that is zero speed of a surface's growth in spite of its being governed by a non-symmetric rule.

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Some physical studies discuss *pinning*, that is zero speed of a surface's growth in spite of its being governed by a rule which is non-symmetric with respect to growth vs. decay. (See e.g. [1], where other references can be found.) It is natural to try to present mathematical counterparts of it. The first idea that comes to mind is to present models where the velocity of growth is exactly zero. Let us call this case exact pinning. Models of this sort are presented inter alia in [2], where Examples 3 and 4 show that speed of growth can be zero even when non-negative noise is added to a symmetric deterministic interaction. However, what is zero in physical observation, does not need to be exact zero in a mathematical model, where the velocity of growth may be positive, but too small to observe. We call this possibility pseudo-pinning. The present paper demonstrates an effect of this sort: velocity of growth decays in a super-polynomial way when parameters of the system tend to zero. In our model components of a surface, which are placed in $\mathbf{Z}$, have real numbers as possible states. They interact in a discrete time in a local deterministic way, in addition to which all the components' states are incremented at every time step by i.i.d. random variables.

In formal terms our process is a sequence of normed measures $P^t\delta_0$, where δ_0 is concentrated in the configuration "all zeros", which we take as the initial condition, and P is a linear operator acting on the set $M(\mathbf{R}^{\mathbf{Z}})$ of normed measures on $\mathbf{R}^{\mathbf{Z}}$ (i.e. on the σ-algebra generated by cylinder sets). Under our assumptions (given below) P commutes with space shifts, whence all the measures $P^t\,\delta_0$ are uniform, that is invariant under space shifts. Given a uniform measure μ, we denote $\mathsf{E}(\mu)$ the expectation of a component. Under our assumptions $\mathsf{E}(P^t\,\delta_0)$ is finite and is a non-decreasing function of t. Our main concern is to estimate the velocity of growth which we define as

$$\mathrm{Vel}(P) = \lim_{t\to\infty} \frac{\mathsf{E}(P^t\,\delta_0)}{t}. \tag{1.1}$$

This limit exists under our assumptions.

We call an operator random if it acts on $M(\mathbf{R}^{\mathbf{Z}})$ and deterministic if it acts on $\mathbf{R}^{\mathbf{Z}}$. Given real parameters $m > 0$ and $\beta \in [0,\ 1]$, we denote $R_{m,\beta}$ the random operator transforming any $x \in \mathbf{R}^{\mathbf{Z}}$ into a product measure, according to which

$$y(s) = \begin{cases} x(s) + m & \text{with probability } \beta, \\ x(s) & \text{with probability } 1-\beta. \end{cases}$$

Given a real function F of three real arguments, we obtain a deterministic operator D transforming any $x \in \mathbf{R}^{\mathbf{Z}}$ into $y \in \mathbf{R}^{\mathbf{Z}}$, where

$$\forall\ s \in \mathbf{Z} : y(s) = F(x(s-1),\ x(s),\ x(s+1)).$$

Actually we are interested in velocities of compositions $R_{m,\beta}D$. If

$$F(x,\ y,\ z) = \min(x,\ y,\ z),$$

then we denote the corresponding deterministic operator D_0. In this case the composition $R_{1,\beta}D_0$ is a well-known percolation operator, for which the possibility of exact pinning has been proved, namely there is a critical value $\beta^* \in (1/8,\ 1/2)$ such that the velocity of $R_{1,\beta}D_0$ is zero for $\beta < \beta^*$ and positive for $\beta > \beta^*$ [2, 3]. We consider a more general operator D_α, into which some 'inertia' is added. Its function F is

$$F(x,\ y,\ z) = \alpha y + (1-\alpha)\min(x,\ y,\ z).$$

Our purpose is to estimate velocity of the composition $R_{m,\beta}D_\alpha$. Let us list some simple facts about it. First, this velocity is proportional to m. (This is true whenever $F(kx,\ ky,\ kz) \equiv k \cdot F(x,\ y,\ z)$.) So it is sufficient to consider the case $m = 1$. $\mathrm{Vel}(R_{1,\beta}D_\alpha)$ does not decrease when α or β increases. In the case $\alpha = 0$ our operator turns into the percolation operator, mentioned above. If $\alpha = 1$, $\mathrm{Vel}(R_{1,\beta}D_\alpha) = \beta$. Therefore $\mathrm{Vel}(R_{1,\beta}D_\alpha) \le \beta$ for any α. Finally $\mathrm{Vel}(R_{1,\beta}D_\alpha) = 0$ if $\beta = 0$.

In computer simulations done by the author the observed velocity was so small for small but positive values of α and β (say, $\alpha = \beta = 0.1$) that its difference from zero could be attributed to the usual deficienciess of computer modeling (say, finiteness of the space, of number of realizations, non-randomness of the "random" numbers etc). However, in fact the velocity is positive whenever $\alpha > 0$ and $\beta > 0$, but tends to zero very fast when $\alpha,\ \beta \to 0$. The following theorem, which is our main result, estimates it:

Theorem. For any $\alpha_0 < 1$ there are positive constants $C_1,\ C_2,\ C_3,\ C_4$ and β_0 such that

$$C_1 \cdot \alpha^{C_2 \ln(1/\beta)/\beta} \le \mathrm{Vel}(R_{1,\beta}D_\alpha) \le C_3 \cdot \alpha^{C_4/\beta} \tag{1.2}$$

for all $0 \le \alpha < \alpha_0$ and $0 < \beta < \beta_0$.

In our proofs we use a partial order on $\mathbf{R}^{\mathbf{Z}}$ defined by the rule $x \prec y \Leftrightarrow \forall\ s :$ $x(s) \le y(s)$. Call a real function f on $\mathbf{R}^{\mathbf{Z}}$ monotonic if $x \prec y \Rightarrow f(x) \le f(y)$. Given $\mu \in M(\mathbf{R}^{\mathbf{Z}})$, denote $E(f \mid \mu)$ the expectation of f. Given $\mu,\ \mu' \in M(\mathbf{R}^{\mathbf{Z}})$, we say that $\mu \prec \mu'$ if $E(f \mid \mu) \le E(f \mid \mu')$ for any monotonic f. Say that a random operator P is monotonic if $\mu \prec \mu' \Rightarrow P\mu \prec P\mu'$.

It is evident that $R_{m,\beta}$ and D_α are always monotonic. Therefore all the components of $(R_{1,\beta}D_\alpha)^t\ \delta_0$ are non-negative a.s. Therefore

$$\mathsf{E}((R_{1,\beta}D_\alpha)^{t+u}\ \delta_0) \ge \mathsf{E}((R_{1,\beta}D_\alpha)^t\ \delta_0) + \mathsf{E}((R_{1,\beta}D_\alpha)^u\ \delta_0)$$

for all natural t and u. Therefore $\mathsf{E}((R_{1,\beta}D_\alpha)^t\delta_0)/t$ is a non-decreasing function of t, whence the limit (1.1) exists in our case. (This argument appliess whenever F is monotonic and $F(0,0,0) = 0$.) Now to prove the theorem.

Proof of the left inequality in (1.2).

For any natural n let us define a deterministic operator $Q_n : \mathbf{R}^Z \to \mathbf{R}^Z$ by the rule $y(s) = \min(x(r) : ns \le r < n(s+1))$. Let us prove that

$$Q_n(R_{1,\beta}D_\alpha)^{nt} \succ (R_{m,\gamma}D_0)^t Q_n \tag{1.3}$$

for all natural t and n, where $m = \alpha^n$ and $\gamma = (1-(1-\beta)^n)^n$. For $t = 0$ this is evident. Then we argue by induction, but first prove that

$$Q_n(R_{1,\beta}D_\alpha)^n \succ R_{m,\gamma}D_0\, Q_n. \tag{1.4}$$

It is evident that $Q_n D_\alpha^n \succ D_0 Q_n$, whence $R_{m,\gamma}Q_n D_\alpha^n \succ R_{m,\gamma}D_0 Q_n$. Thus to prove (1.4) it remains to prove that

$$Q_n(R_{1,\beta}D_\alpha)^n \succ R_{m,\gamma}Q_n D_\alpha^n. \tag{1.5}$$

Action of $R_{m,\beta}$ can be represented using a product-measure on $\{0,1\}^Z$, according to which every component $h(s)$ equals 1 with probability β and 0 with probability $1-\beta$. For any $x \in \mathbf{R}^Z$ the measure $R_{m,\beta}\, x$ is induced by this product-measure with the map $y(s) = x(s) + m \cdot h(s)$. Now, to prove (1.5) it is sufficient to apply both sides to a measure concentrated in one configuration x and couple them as follows:

$$g(s) = \max(h_t(r) : ns \le r < n(s+1),\ 1 \le t \le n),$$

where $g(s)$ serve $R_{m,\gamma}$ and $h(s)$ serve $R_{1,\beta}$ in the way described above. If $h_t(s) \equiv 0$, our statement is evident. Now let us see how the components of both sides increase if some $h_t(s) > 0$. For any integer $q > 0$, r, s we denote Impact$(q,\ r,\ s)$ the minimal amount by which the s-th component of $D_\alpha^q\, x$ increases if $x(r)$ increases by 1, all the other components of x remaining unchanged. It is easy to see that

$$\forall\ q,\ r,\ s : \text{Impact}(q,\ r,\ s) \ge \begin{cases} \alpha^q & \text{if } r = s, \\ 0 & \text{otherwise}. \end{cases} \tag{1.6}$$

Due to (1.6), the event

$$\{\forall\ r \in [ns,\ n(s+1)-1]\ \ \exists\, t \in [1,\ n]\ :\ h_t(r) = 1\} \tag{1.7}$$

guarantees that the r-th component of $(R_{1,\beta}D_\alpha)^n\, x$ is greater than the r-th component of $D_\alpha^n\, x$ at least by α^n for all r in the range $ns \le r < n(s+1)$. Therefore the s-th component of $Q_n(R_{1,\beta}D_\alpha)^n\, x$ is greater than the s-th component of $Q_n D_\alpha^n\, x$ at least by α^n. The probability of event (1.7) is $(1-(1-\beta)^n)^n$. Thus (1.5) is proved. Hence (1.4) follows, using which we can prove (1.3) by induction.

Now to prove the left inequality in (1.2). For any positive β the expression $(1-(1-\beta)^n)^n$ tends to 1 when n tends to infinity. More than that, there are positive C and β_0 such that $\gamma = (1-(1-\beta)^n)^n > \beta^*$ for $\beta < \beta_0$ and $n > C\ln(1/\beta)/\beta$. The minimal n satisfying this condition does not exceed $\text{const}\cdot\ln(1/\beta)/\beta$. Remember that $R_{1,\gamma}D_0$ is an operator of a percolation process. Since $\gamma > \beta^*$, this process grows, whence

$$L := \lim_{t\to\infty} \frac{E((R_{1,\gamma}D_0)^t\,\delta_0)}{t} = \text{const} > 0.$$

Therefore

$$\lim_{t\to\infty} \frac{\mathsf{E}(Q_n(R_{1,\beta}D_\alpha)^{nt}\,\delta_0)}{t} \geq \lim_{t\to\infty} \frac{\mathsf{E}((R_{m,\gamma}D_0)^t\,\delta_0)}{t} = L\cdot m = L\cdot\alpha^n.$$

Therefore

$$\begin{aligned}\lim_{t\to\infty} \frac{\mathsf{E}((R_{1,\beta}D_\alpha)^t\,\delta_0)}{t} &= \lim_{t\to\infty} \frac{\mathsf{E}((R_{1,\beta}D_\alpha)^{nt}\,\delta_0)}{nt} \\ &\geq \frac{L\cdot\alpha^n}{n} \geq \text{const}\cdot\frac{\beta}{\ln(1/\beta)}\cdot\alpha^{\text{const}\cdot\ln(1/\beta)/\beta},\end{aligned}$$

where the constants are positive. Since α is separated from 1, we may compensate the factor $\beta/\ln(1/\beta)$ by adjusting the constants. Thus the left inequality in (1.2) is proved. □

Proof of the right inequality in (1.2).

Denote $M := M(\{0,1\}^Z)$ the set of normed measures on $\{0,1\}^Z$. For any $\varepsilon \in [0,1]$ define $M_\varepsilon \subset M$ as follows: $\mu \in M$ belongs to M_ε if

$$\mu(x(s)=1 \quad \text{for all} \quad s\in S) \leq \varepsilon^{|S|} \quad \text{for any finite} \quad S\subset Z, \tag{1.8}$$

where $|S|$ is the cardinality of S. Let us prove that for any α, $\beta \in [0,\ 1]$ and any natural n there is $\mu \in M_\varepsilon$, where $\varepsilon = 2(n\beta)^{1/3}$, such that

$$(R_{1,\beta}D_\alpha)^n \prec \Gamma_{n,\mu}\Delta_{n,\alpha}, \tag{1.9}$$

where $\Delta_{n,\alpha} : \mathbf{R}^Z \to \mathbf{R}^Z$ is a deterministic operator defined by the formula

$$y(s) = \begin{cases} x(s) + n\cdot\alpha^n & \text{if } \min(x(s-1),\ x(s+1)) > x(s) - n, \\ x(s) + n\cdot\alpha^n - n & \text{otherwise}, \end{cases}$$

and $\Gamma_{n,\mu} : M \to M$ is a random operator defined by the formula $y(s) = x(s) + n\cdot g(s)$ where $g(s)$ are random variables, distributed according to μ.

First let us prove that $D_\alpha^n \prec \Delta_{n,\alpha}$, applying both to an arbitrary configuration x and considering two cases:

- If $\min(x(s-1),\ x(s+1)) > x(s) - n$, the s-th component of $\Delta_{n,\alpha}x$ is greater than $x(s)$, which is always not less than the s-th component of $D_\alpha^n\,x$.

- If $\min(x(s-1),\ x(s+1)) \le x(s) - n$, the s-th component of $\Delta_{n,\alpha} x$ is $x(s) + n \cdot \alpha^n - n$, which is not less than the s-th component of $D_\alpha^n\, x$.

Thus $D_\alpha^n \prec \Delta_{n,\alpha}$, whence $\Gamma_{n,\mu} D_\alpha^n \prec \Gamma_{n,\mu}\Delta_{n,\alpha}$. To prove (1.9) it remains to find $\mu \in M_\varepsilon$ such that

$$(R_{1,\beta} D_\alpha)^n \prec \Gamma_{n,\mu} D_\alpha^n. \tag{1.10}$$

For this purpose we represent different applications of $R_{1,\beta}$ using i.i.d. random variables $h_t(s)$, everyone of which equals 1 with probability β and 0 with probability $1-\beta$ and define $\mu \in M$ as induced by them with the map

$$g(s) = \max(h_t(r)\ :\ s-1 \le r \le s+1,\ 1 \le t \le n). \tag{1.11}$$

It is easy to prove that $\mu \in M_\varepsilon$, where $\varepsilon = 2(n\beta)^{1/3}$. Actually the formula (1.11) defines a coupling of the two sides of (1.10), and thereby allows to prove it. Hence (1.9) follows.

Now define $\Delta' : \mathbf{R}^\mathbf{Z} \to \mathbf{R}^\mathbf{Z}$ by the formula

$$y(s) = \begin{cases} x(s) & \text{if } \min(x(s-1),\ x(s+1)) > x(s) - n, \\ x(s) - n & \text{otherwise}. \end{cases}$$

Let us prove that there is $\varepsilon^* > 0$ such that

$$\mathrm{Vel}(\Gamma_{n,\mu}\Delta') = 0 \tag{1.12}$$

for any n and $\mu \in M_\varepsilon$, where $\varepsilon < \varepsilon^*$. Notice that in this case all the components are integer multiples of n a.s. So we can divide all values of components by n and obtain a process on $\mathbf{Z}^\mathbf{Z}$ rather than $\mathbf{R}^\mathbf{Z}$. First assume that μ is a product-measure, that is all $g(s)$ are i.i.d. and each equals 1 with probability ε and 0 with probability $1-\varepsilon$. This case is covered by the Theorem 2 in [2], whence (1.12) follows. However, it is clear from the proof of this theorem, given in [3], that all that is needed is the condition (1.8). Thus (1.12) is proved.

Now to prove the right inequality in (1.2). Notice that the only difference between Δ' and $\Delta_{n,\alpha}$ is the term $n \cdot \alpha^n$. Hence it follows from (1.9) and (1.12) that $\mathrm{Vel}(R_{1,\beta} D_\alpha)$ does not exceed α^n as long as $2(n\beta)^{1/3} < \varepsilon^*$. Let us choose n the greatest for which still $2(n\beta)^{1/3} < \varepsilon^*$. Thus chosen n is greater than C/β with an appropriate $C > 0$ for all $\beta < \beta_0$, where β_0 is an appropriate positive constant. Hence the right inequality in (1.2) follows. □

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