NONERGODIC MULTIDIMENSIONAL SYSTEMS
OF AUTOMATA

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Identical stochastic automata having a finite number of states are positioned at all points of a d-dimensional integer-valued space. At any instant of discrete time each automaton can go to any one of its states with never-vanishing probabilities depending on its own states and those of a finite number of its "neighbors" at the preceding instant. A system of this type is synthesized which is capable of "remembering" its initial state for an infinitely long time when the system commences operation in one of n distinct states of the type "all automata are in state k," where 1 ≤ k ≤ n.

1. Introduction

We propose to investigate a Markov chain having a continuum of states and describing the behavior of an infinite system of stochastic automata. All automata are identical. They are presumed to be situated at the nodes of the d-dimensional integer-valued lattice \( \mathbb{Z}^d \), where they are enumerated by index vectors \( i \in \mathbb{Z}^d \) (i is any d-dimensional integer-valued vector). The state of each automaton assumes a finite set of values \( M_m = \{0, 1, \ldots, m\} \). The time \( t \) is discrete. The state \( x_{it}^t \) of the i-th automaton at time \( t \) depends probabilistically on the states of \( r \) designated automata, known as its "neighbors," at time \( t-1 \), i.e., \( x_{i}^{t-1} = b \) with probability \( \varphi^b(a_1, \ldots, a_r) \) if

\[
x_{i+\nu_i}^{t-1} = a_i, ..., x_{i+\nu_r}^{t-1} = a_r, \text{where } a_1, ..., a_r, b \in M_m, \quad \sum_b \varphi^b = 1.
\]

The set of vectors \( V_1, ..., V_r \) and the function \( \varphi^b \) is the same for all automata. The set of indices \( i + V_1, ..., i + V_r \) is denoted by \( U(i) \). If the states of all automata at time \( t-1 \) are given, the states of the automata at time \( t \) are independent stochastic variables.

The foregoing description of the operation of automata determines a linear operator \( P_\varphi \) in measure space on the set \( X_m = M_m^d \), where the \( \sigma \)-algebra is generated by cylindrical sets. Let \( a \in X_m \), i.e., \( a = (a_i), a_i \in M_m, i \in \mathbb{Z}^d \). We denote by \( \delta_a \) the measure on \( X_m \) concentrated at \( a \). The operator \( P_\varphi \) maps \( \delta_a \) into a measure \( \delta_{a} P_\varphi \) in which all automaton states \( x_{i}, i \in \mathbb{Z}^d \) are independent and \( x_i = b \) with probability \( \varphi^b(a_{i+V_1}, ..., a_{i+V_r}) \). The outcome of an application of \( P_\varphi \) to any measure \( \mu \) is defined as

\[
\mu P_\varphi(x; z = b_i, \text{ for all } i \in I) = \sum_{\{a_i\} \in U(I)} \mu(x; z = a_i, \text{ for all } j \in U(I) \prod_{i \in I} \varphi^b(a_{i+v_r}, ..., a_{i+v_r}),
\]

where \( x = (x_i), x \in X_m \) and \( U(I) = \bigcup_i U(i) \).

We refer to operators \( P_\varphi \) of the type described above as operators of type (1). A measure \( \mu \) is said to be invariant for a given operator \( P_\varphi \) if \( \mu = \mu P_\varphi \). An operator \( P_\varphi \) is called ergodic if it has only one, up to a multiplier, invariant measure (there is always at least one); otherwise it is called nonergodic.

The fundamental result of the article is proof of the nonergodicity of a certain class of operators \( P_\varphi \) of the given type and the construction for any positive integer \( n \) of operators \( P_\varphi \) having at least \( n \) distinct


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Invariant measures $\mu_1, \ldots, \mu_n$. Here $n$ can be any positive integer compatible with dimensions $d = 2$ or more, but the number $m + 1$ of automaton states must be made greater than or equal to $n$.

Nonergodic operators of the type described above or analogous thereto have been constructed in several papers [1-6]. In [1-5], however, the authors rely heavily on the fact that certain values of the functions $\varphi^b$ are equal to zero, and nonergodicity is associated with the existence of "degenerate" invariant measures equal to zero on certain cylindrical sets. Dobrushin [6] has demonstrated a technique for the transformation of Ising models into Markov chains of a form analogous to that considered here, but only for continuous time $t$. A phase transition in the Ising models guarantees nonergodicity of the chains in this situation.

In our work the functions $\varphi^b(a_1, \ldots, a_T)$ are positive for all $a_1, \ldots, a_T, b$, so that the values of the invariant measures on all cylindrical sets are also positive. Moreover, the conditions imposed on $\varphi^b(a_1, \ldots, a_T)$ for the proof of the fundamental theorem constitute inequalities and are therefore satisfied by an entire domain in the space of sets of values of $\varphi^b(a_1, \ldots, a_T)$.

The operators $P_\varphi$ constructed below depend on a parameter $\varepsilon$, $0 \leq \varepsilon \leq 1$, and their nonergodicity is proved only for sufficiently small values of $\varepsilon$. For $\varepsilon = 0$ the operator $P_\varphi$ becomes deterministic, i.e., the functions $\varphi^b$ assume only the values 0 or 1. For $\varepsilon = 0$ the invariant measures $\mu_1, \ldots, \mu_n$ go over to measure $\delta_1, \ldots, \delta_n$ concentrated in states of the type "all $x_i$ are equal to $k$.''

Every operator $P_\varphi$ discussed below is the superposition of an operator $P_f$ and an operator $S_\alpha$ ($P_f$ is applied to the measure first, and then $S_\alpha$). Here $P_f$ is a deterministic operator, i.e., $P_f$ maps measure $\delta_a$ into $\delta_a'$, where the state $a' = (a'_i)$ is the following function of the state $a = (a_i)$, $i \in \mathbb{Z}^d$.

$$a'_i = f(a_1, \ldots, a_m).$$

We refer to operators $P_f$ of the type just described as operators of type (2). Here $f$ is a deterministic function of $r$ arguments $f: M^r \rightarrow M^r$, and is the same for all $i \in \mathbb{Z}^d$.

The operators $S_\alpha$ is described by a stochastic matrix of order $m + 1: \alpha = \{\alpha_{k,l}\}$, $1 \leq k, l \leq m + 1$. Broadly, its action may be summed up as taking every automaton found in a state $k$ ($1 \leq k \leq m + 1$) into state $l$ with probability $\alpha_{k,l}$ independently of all other automata. In precise terms, the operator $S_\alpha$ maps every measure $\delta_a$ into a measure in which all $x_k$ are independent and $x_l = b$ with probability $\alpha_{a_i,b}$.

It is easily verified that the product $P_\varphi = P_f S_\alpha$ is an operator of type (1), where

$$\varphi^b(a_1, \ldots, a_T) = \alpha_{f(a_1), \ldots, f(a_T), b}.$$  

We denote by $\beta(\varepsilon)$ the matrix of order $m + 1$ formed by elements of the type

$$\beta_{k,l} = \begin{cases} e/\varepsilon & \text{if } k \neq l, \\ 1-\varepsilon & \text{if } k = l, \end{cases}$$

where $0 \leq k, l \leq m, 0 \leq \varepsilon \leq 1$. As $\varepsilon \to 0$ the matrix $\beta(\varepsilon) \to E$ and the operator $S_\beta(\varepsilon)$ tends to the identity operator. Following is our fundamental result.

THEOREM. For any positive integer $n$, it is possible to construct an operator $P_f$ such that for sufficiently small (but positive) values of $\varepsilon$ the operator $P_\varphi = P_f S_\beta(\varepsilon)$ has at least $n$ distinct invariant measures.

The fundamental lemma proved in §2 below is central to the proof of the theorem. In §3 we construct the operator $P_f$ and complete the proof of the theorem. In all the proofs the matrix $\beta(\varepsilon)$ can be replaced by any matrix $\alpha$ in which $\alpha_{k,l} \leq \varepsilon/m$, as long as $k \neq l$.

What we actually do in §3 is to construct an operator $P_f$ having the property designated by the formula

$$\lim \max \sup_{\varepsilon \to 0} \delta_k(P_f S_{\varepsilon}(x,z,x,k)) = 0.$$  

Relation (3) implies that for sufficiently small values of $\varepsilon$ the measures $\delta_k$ are "stable" in the sense that they change very little after any number of applications of the operator $P_\varphi = P_f S_\beta(\varepsilon)$. Hence, applying the fixed-point theorem, we readily infer the existence of invariant measures $\mu_0, \ldots, \mu_m$ that tend weakly to $\delta_0, \ldots, \delta_m$ as $\varepsilon \to 0$ and are therefore distinct for sufficiently small $\varepsilon > 0$.
2. The Fundamental Lemma

We limit the discussion of this section to the case \( m = 1 \), i.e., to automata having two states: 0 and 1. Let the operator \( S_{\beta^0(e)} \) be given by the following matrix \( \beta^0(e) \) of order two:

\[
\beta_{i,0}^{0,0} = 1 - e, \quad \beta_{i,1}^{0,0} = e, \quad \beta_{i,0}^{0,1} = 0, \quad \beta_{i,1}^{0,1} = 1.
\]

Broadly, the action of \( S_{\beta^0(e)} \) may be summed up as leaving all automata found in the state 1 in that state and taking all automata found in the state 0, each independently of the other, into the state 1 with probabilities equal to \( e \).

**LEMMA.** Let an operator \( P_f \) of type (2) be specified for the case \( m = 1 \) by the set of vectors \( U(0) = \{V_1, \ldots, V_r\} \) and the function \( f(\sigma_1, \ldots, \sigma_r) \), which is Boolean in the given case. Let a real number \( c \) and a nondegenerate linear functional \( L(l) \) on \( Z^d \) exist such that if

\[
\begin{align*}
(x_r &= 0 \text{ for all } V_r \in U(0), \text{ such that } L(V_r) \leq c, \text{ or } \\
(x_r &= 0 \text{ for all } V_r \in U(0), \text{ such that } L(V_r) > c),
\end{align*}
\]

then \( f(x_1, \ldots, x_r) = 0 \). For all \( t = 0, 1, 2, \ldots \) and all \( i \in Z^d \), there exists an upper bound of the form

\[
\delta_{0} P_{f}(x : x_{i} = 1) \leq a(e),
\]

where the function \( a(e) \) is defined for \( 0 \leq e < e_0 \) and \( \lim_{e \to 0} a(e) = 0 \).

Here \( P_{f} = P_{f} S_{\beta^0(e)} \), and \( \delta_{0} \) is a measure concentrated in the "all-zeros" state.

**Proof of the Lemma.** Together with the function \( f \) we consider a function \( f' \) such that to have \( f'(x_1, \ldots, x_r) = 0 \) condition (4) is not only sufficient, but necessary as well. Clearly, \( f' \equiv f \) (for any set of values of the arguments). Consequently, \( P_{f} > P_{f'} \) in the sense of Mityushin [4], and it suffices to prove inequality (3) for \( P_{f'} = P_{f} S_{\beta^0(e)} \). Clearly, \( f'(x_1, \ldots, x_r) \) can be represented in the form

\[
f'(x_1, \ldots, x_r) = \bigvee_{k : L(V_k) < c} x_r \bigvee_{k : L(V_k) > c} x_r \bigvee_{k : L(V_k) = c} x_r.
\]

We denote

\[
c' = \min_{L(V_r) > c} L(V_r).
\]

We rewrite (6), using the \( c' \) just defined and enumerating all pair products in (6) in sequence by numbers from 1 to \( q \), \( q \) being equal to the product of the number of values of \( k \), where \( L(V_k) \leq c \), by the number of values of \( l \), where \( L(V_l) > c \):

\[
f'(x_1, \ldots, x_r) = \bigvee_{k \leq q} x_r x_r',
\]

where \( L(V_k) \leq c \), \( L(V_k') \geq c' \) for all \( k \), \( 1 \leq k \leq q \). Obviously, the list of vectors \( V_1', \ldots, V_q' \) consists of the vectors \( V_k \) for which \( L(V_k) \leq c \) with repetitions, and the list of vectors \( V_1'', \ldots, V_q'' \) consists of the vectors \( V_l \) for which \( L(V_l) \geq c' \) with repetitions.

We represent the operator \( S_{\beta^0(e)} \) as follows. We introduce stochastic variables \( \gamma_i \), which are independent of one another and are equal to 0 with probability \( 1 - e \) and to 1 with probability \( e \). Then the measure \( \delta_{\gamma} S_{\beta^0(e)} \) is induced by the distribution of \( \gamma_i \) under the map

\[
x_i = a_i \vee \gamma_i, \quad i \in Z^d.
\]

At the initial time \( t = T \) (\( T \) being a positive integer) let the measure \( \delta_{x^T} \) be given. Let the operator \( P_{f} \) then be applied to it repeatedly. We denote by \( \gamma'_i \) the variables \( \gamma_i \) participating in an application of \( P_{f} \) that maps the measure at time \( t-1 \) into the measure at time \( t \). Of course, all the \( \gamma'_i \) are mutually independent. Clearly, the state \( x^T \) of the automaton \( x^T \) at time \( t = 0 \) is a deterministic function of a finite number of \( \gamma_i' \) and \( x^T' \). We denote that function by \( F_T \):

\[
x_i = F_T((\gamma'_i, |i| < -RT, -T < t \leq 0), (\gamma'_i, |i| < RT)).
\]

where \( |i| \) is the sum of the moduli of the components \( i \in Z^d \), \( R = \max_{1 \leq i < r} |V_{i}| \). At \( T = 0 \) the function \( F_T \) goes over to \( F_0 \), which is equal to
For $T \geq 1$ the expression for $F_T$ is obtained by substitution of the expressions

$$y_i^{T+1} \lor f'(z_{i_1}^{-T}, \ldots, z_{i_q}^{-T})$$

for all $x_i^{-T+1}$ contained in the expression for $F_{T-1}$. It is obvious that for all $T$ the function $F_T$ is monotone Boolean. It is therefore representable (in several ways) as the disjunction of conjunction-terms of its arguments. We define one such representation

$$F_T = \bigvee_{(i,t) \in G} \bigwedge_{(i,t) \in G} \gamma_i^{T+1}, \text{where } t \geq -T, \text{ or } x_i^{-t}$$

inductively on $T$. Here $G$ is the set of pairs $(i, t)$ enumerating those $\gamma_i^t$ or $x_i^{-T}$ occurring in the given term, and $H_T$ is the set of sets $G$ corresponding to all terms of the disjunction.

Let us suppose that an expression of the form (8) has already been obtained for $F_{T-1}$. To obtain an expression of the form (8) for $F_T$ we need to replace every $x_i^{-T+1}$ in the expression (8) for $F_{T-1}$ by

$$\gamma_i^{T+1} \lor \bigvee_{k=1}^q x_{i_0'v_k}^{T'}, x_{i_0'v_k}^{-T'}$$

and develop the parentheses. This defines an expression of the form (8) inductively (we do not know whether it is a disjunctive normal form).

**Definition.** A cycle $Q$ is a combination formed by a finite sequence of pairs of the form $(i, t) \in \mathbb{Z}^{d+1}$ (some of the pairs can be identical) and a set $G_Q$ containing certain pairs entering into that sequence:

$$Q = \{(i_k, t_k), 0 \leq k \leq S_Q - 1\}, G_Q, \quad (9)$$

where the following conditions are satisfied:

a) $(i_0, t_0) = (0, 0)$;

b) all pairs in $G_Q$ enter into the sequence in (9).

We define

$$\Delta_k = \begin{cases} (i_{k+1}, t_{k+1}) - (i_k, t_k), & \text{if } 0 \leq k \leq S_Q - 2, \\
(i_0, t_0) - (i_{S_Q - 1}, t_{S_Q - 1}), & \text{if } k = S_Q - 1; \end{cases}$$

c) for any $k, 0 \leq k \leq S_Q - 1$ the difference vector $\Delta_k$ has one of the alternative forms $(V_{i_0}^t, -1)$, $(V_{i_0}^t - V_{i_0}, 0)$, or $(-V_{i_0}^t, 1)$, where $1 \leq l \leq q$;

d) if $\Delta_k, 0 \leq k \leq S_Q - 1$ has the form $(V_{i_0}^t - V_{i_0}, 0)$, then the pair $(i_k, t_k)$ enters into the set $G_Q$.

**Remark.** Suppose that the sequence involved in a cycle of the form (9) satisfies conditions a) and c).

We estimate the lower bound of the number of indices $0 \leq k \leq S_Q - 1$ for which $\Delta_k$ has the form $(V_{i_0}^t - V_{i_0}, 0)$. We introduce the following linear functional on $\mathbb{Z}^{d+1}$: $L'(i, t) = L(i) + (1/2)(c' + ct)$.

It is readily shown by calculations that $L'(\Delta_k) \leq (1/2)(c - c')$ if $\Delta_k$ has the form $(V_{i_0}^t, -1)$ or $(-V_{i_0}^t, 1)$. We define

$$D = \max_{0 \leq c \leq q} L'(V_{i_0}^t - V_{i_0}, 0).$$

Clearly,

$$\sum_{k=0}^{S_Q-1} \Delta_k = 0,$$

whereupon

$$\sum_{k=0}^{S_Q-1} L'(\Delta_k) = 0.$$

We separate the latter sum into two parts. We assign to the first sum those terms for which $\Delta_k$ has the form $(V_{i_0}^t, -1)$, $1 \leq l \leq q$, and to the second sum all remaining terms. Suppose that the first sum comprises $h$ terms. Then it is not greater than $Dh$. The second sum is not greater than $(1/2)(c - c')(S_Q - h)$. Therefore, $0 \leq Dh + (1/2)(c - c')(S_Q - h)$, whence
We now prove the following statement by induction on \( T \). For every \( G \in H_T \), where \( H_T \) is defined 

\[ h > \frac{c' - c}{2D + c - c} S_q. \]  

(10)

as far as the form (8) is defined, there is a cycle \( Q \) such that \( G = G_Q \). For \( T = 0 \) the set \( H_T \) comprises only one set \( G \), which consists of the single pair \((0, 0)\). For this \( G \) we choose \( S_Q = t \) and the sequence of pairs \((0, 0), (V_1', -V_1', 0), (-V_1', 1)\). (Instead of \( V_1', V_1'' \) we could have chosen \( V_i', V_i'' \) for any \( i, 1 \leq i \leq q \).) Next we go from \( T = 1 \) to \( T \). We have described an algorithm for the 
derivation of the form (8) for \( F_T \) from the form (8) for \( F_{T-1} \). Every term of the form for \( F_T \) is obtained from some term of the form for \( F_{T-1} \) in such a way that every \( x_i^{T+1} \) is replaced either by \( \gamma_i^{T+1} \) or by some product \( x_i + V_i x_i + V_i' \), where \( 1 \leq i \leq q \). By the induction hypothesis the leading term in the form (8) for \( F_{T-1} \) has associated with it a cycle \( Q \) of the form (9). We transform it into the cycle \( Q' \) corresponding to a new term entering into the form (8) for \( F_T \).

We call an index \( k \), \( 0 \leq k \leq S_Q - 1 \) singular (for a given leading term and given new term derived therefrom) if the term \((i_k, t_k)\) of the sequence in \( Q \) meets the following conditions: \( t_k = -T + 1 \); the factor \( x_{i_k}^{t_k} \) is included in the leading term; in transition to the new term the indicated factor is replaced by a product of the form

\[ x_i + V_i x_i + V_i', \quad \text{where} \quad 1 \leq i \leq q. \]  

(11)

The sequence involved in \( Q' \) differs from the sequence \((i_k, t_k)\), \( 0 \leq k \leq S_Q - 1 \) to the extent that after every term \((i_k, t_k)\) thereof with a singular index \( k \) three new terms are inserted into the sequence in the following order:

\[(i_k + V_i', -T), (i_k + V_i'', -T), (i_k + V_i, -T),\]  

(12)

where the index \( i \) coincides with the index \( i \) in (11). The resulting sequence lengthened by three times the number of singular indices is reenumerated by integers beginning with zero.

The set \( G_{Q'} \) is defined as the set of pairs \((i, t)\) enumerating all factors of the new term. It is easily proved that the \( Q' \) so constructed satisfies conditions a) through d).

We now deduce the bound (5), which is the substance of the lemma. We must find an upper bound for the probability that \( x_i^0 = 1 \) when the "all zeros" measure \( \delta_0 \) is specified at time \(-T\). (Obviously, for any \( i \in \mathbb{Z}^d \) the probability that \( x_i^0 = 1 \) is the same as for \( i = 0 \).) In other words, we must find an upper bound for the probability that \( F_T(\{\gamma_i^t, 0 \leq i \leq R, -T < t \leq 0\}, \{x_i^{-T}, 0 \leq i \leq R\}) = 1 \), when all \( x_i^{-T} \) are identically zero, while all \( \gamma_i^t \) are independent and equal to zero with probability \( 1 - \theta \) and to unity with probability \( \theta \). We denote by \( F_T^0(\{\gamma_i^t, 0 \leq i \leq R, -T < t \leq 0\}) \) the function obtained from \( F_T \) by the substitution of zeros for all \( x_i^{-T} \). Clearly,

\[ F_T^0 = \bigvee_{0 \leq r \leq T} \bigwedge_{t=1}^{r} \gamma_i^t, \]  

(13)

where the set \( H_T^0 \) is obtained from \( H_T \) in (8) by the deletion of all sets included therein containing at least one pair of the form \((i, -T)\).

We define a complex \( K \) as a nonvacuous finite set (N-tuple) of cycles

\[ K = (Q_1, \ldots, Q_N). \]  

(14)

We say that a complex \( K \) generates a Boolean function \( F \) of a finite number of variables \( \gamma_i^t \) if

\[ F = \bigvee_{t=1}^{R} \bigwedge_{i=1}^{N} \gamma_i^t. \]  

(15)

Here \( G_{Q_i} \) are the sets entering into the cycles \( Q_i \) forming \( K \). We have proved above that the function \( F_T^0 \) is representable as generated by a certain complex \( K \) of the form (14), where \( Q_1, \ldots, Q_N \) are the cycles corresponding to all sets \( G \) entering into \( H_T^0 \) in (13). (Since \( T \) is now fixed, we do not attach an index to it.)

We transform the complex \( K \) into another complex \( K' = (Q_1', \ldots, Q_N') \) in such a way as to meet conditions a) through d) below. We denote by \( F_T' \) the function generated by \( K' \).
a) \( F_T \leq F_\bar{T} \) (on all sets of values of \( \gamma_i^l \));
b) for each cycle \( Q_i^l, 1 \leq l \leq N', \) all terms of its member sequence are distinct;
c) all sets \( G_{Q_i^l} \) entering into the cycles \( Q_i^l, 1 \leq l \leq N' \) are not subsets, any one of any other. (In particular, therefore, all \( G_{Q_i^l} \) are distinct, so certainly all \( Q_i^l \) are distinct);
d) the sequence entering into each cycle \( Q_i^l, 1 \leq l \leq N' \) contains at least three terms.

We construct in succession complexes \( K^0, K^1, K^2, \ldots \), where \( K^0 = K \), until we obtain a complex \( K^n \) that can be adopted as \( K' \). The complexes \( K^n \) are formed inductively. Suppose that the following complex has already been constructed:

\[
K^n = (Q_i^1, \ldots, Q_i^{n+1}).
\]

If \( K^n \) satisfies conditions b) and c), we can adopt it as \( K' \). Incidentally, conditions a) and d) are satisfied by all \( K^n, n = 0, 1, 2, \ldots \).

Let \( K^n \) not satisfy condition b). Then in one cycle \( Q_i^n \) of the form \( Q_i^n = (l_k, t_k), 0 \leq k \leq S_{Q_i^n} - 1 \), \( G_{Q_i^n} \) two pairs entering into the member sequence are identical: \( (l_{k_1}, t_{k_1}) = (l_{k_2}, t_{k_2}) \), where \( k_1 < k_2 \).

In this case we delete from the sequence in \( Q_i^n \) all pairs \( (l_k, t_k) \), \( k_1 < k < k_2 \). We enumerate the remaining pairs in succession by integers beginning with zero. We also eliminate from the set \( G_{Q_i^n} \) those elements \( (i, t) \) that are now included in the sequence. Clearly, we obtain as a result another cycle, which we call \( (Q_i^n)' \). Replacing the cycle \( Q_i^n \) in \( K^n \) by \( (Q_i^n)' \), we obtain a new complex \( K^{n+1} \).

Let \( K^n \) not satisfy condition c). Then for a certain pair of cycles \( Q_i^n, Q_j^n \) entering into \( K^n \) the set \( G_{Q_i^n} \) is a subset (possibly not a proper subset) of the set \( G_{Q_j^n} \). In this case the list of cycles defining the complex \( K^{n+1} \) is obtained from the list (16) by deletion of the cycle \( Q_i^n \).

It is easily verified that after a finite number of such reconstructions of the complex \( K \) a complex satisfying conditions a) through d) is obtained, which is then taken as \( K' \). From condition a) and the definition of \( F_T \) we have

\[
Pr(F_{r^T} = 1) = Pr\left( \bigvee_{1 \leq i \leq N'} \bigwedge_{(l_i, t_i) \in G_{Q_i^l}} \gamma_i^l = 1 \right)
\leq \sum_{i=1}^{N'} Pr\left( \bigwedge_{(l_i, t_i) \in G_{Q_i^l}} \gamma_i^l = 1 \right) - \sum_{i=1}^{N'} e^{10q_i^l 1},
\]

where \( |G_i| \) is the number of elements in \( G_i \).

From condition b) for the complex \( K' \) and condition d) in the definition of a cycle the number \( |G_{Q_i^n}| \) of elements in \( G_{Q_i^n} \) is not less than the number of difference vectors \( \Delta_k \), \( 0 \leq k \leq S_{Q_i^n} - 1 \) in the \( Q_i^l \) sequence of the form \( (V_j^i - V_j^i, 0) \), where \( 1 \leq j \leq q \). We therefore obtain from relation (10)

\[
|G_{Q_i^n}| \geq \frac{c'-c}{2D+c'-c} S_{Q_i^n}.
\]

We separate the last sum in (17) into a sum of sums. We assign to the \( j \)-th sum those terms for which the number \( S_{Q_i^n} \) of the elements in the \( Q_i^l \) sequence is equal to \( j \). By condition d) for the complex \( K' \) only the sets of terms of sums for \( j \geq 3 \) are nonvacuous. Consequently, taking (18) into account, we obtain

\[
\sum_{1 \leq i \leq N'} e^{10q_i^l 1} = \sum_{j=3}^{\infty} \sum_{S_{Q_i^n}=j} e^{10q_i^l 1} \leq \sum_{j=3}^{\infty} \sum_{S_{Q_i^n}=j} \left( e^{\frac{c'-c}{2D+c'-c}} \right)^j.
\]
From condition c) for the complex \( K' \) we know that all cycles in \( K' \) are distinct. Invoking conditions a), b), and c) in the definition of a cycle, we readily verify that there are at most \((6q)^j\) distinct cycles whose member sequences contain \( j \) terms. Therefore, the last expression in (19) is not greater than

\[
\sum_{j=1}^{\infty} (6q)^j \left( \frac{s^j}{6q^{2j+2}} \right)^j = (6q s^j 6q^{2j+2})^{1/(1-6q 2j+2)} = o(\varepsilon). \tag{20}
\]

This equation serves as a definition of \( o(\varepsilon) \). As \( \varepsilon \to 0 \) it tends to zero, whereupon we arrive at the conclusion of the lemma.

13. Proof of the Fundamental Theorem

Let a positive integer \( n \) be given. It is required to construct an operator \( P_0 = P_0 S_0(\varepsilon) \) having at least \( n \) distinct invariant measures. We set \( m = n \). We determine \( f \). To do so we choose \( m + 1 \) linear functionals \( L_k, 1 \leq k \leq m + 1 \), on \( Z^d \). All that is required is that the functionals \( L_k \) be nondegenerate and pairwise nonproportional; otherwise they can be arbitrary. We define a finite set \( U(0) = \{V_1, \ldots, V_r\} \) so that it has a nonvacuous intersection with each of the four quarter lattices into which \( \mathbb{Z}^d \) is divided by each pair of hyperplanes \( L_k = 0, L_l = 0, 1 \leq k \neq l \leq m + 1 \). It is sufficient, for example, to draw through the point 0 in \( \mathbb{Z}^d \) a two-dimensional rational plane on which each of the functionals \( L_k \) is not constant and to form the set \( U(0) \) of \( (m+1) \) integer-valued points belonging to that plane, one per node, at which that plane intersects the hyperplanes \( L_k = 0, 1 \leq k \leq m + 1 \). Then the following definition of \( f(x_{V_1}, \ldots, x_{V_r}) \) is noncontradictory:

\[
f(x_{V_1}, \ldots, x_{V_r}) = k, \quad \text{if } \begin{cases} x_{V_i} = k & \text{for all } V_i \in U(0) \text{ such that } L_k(V_i) < 0, \\ x_{V_i} = k & \text{for all } V_i \in U(0) \text{ such that } L_k(V_i) > 0. \end{cases} \tag{21}
\]

For the case in which condition (21) is not satisfied for any \( k \), the definition of the function is augmented in any way necessary. We are required to prove that condition (3) is satisfied. We show that the following specific condition holds, which clearly guarantees (3):

\[
\delta_0 P_0(x : x_i \neq k) = o(\varepsilon), \tag{22}
\]

where the function \( o(\varepsilon) \) is defined in (20).

We fix \( k \) from the outset, \( 0 \leq k \leq m \). We use the fundamental lemma proved in §2. For convenience we denote by \( Y \) the space \( X_1 \) and let the points of \( Y \) be denoted by \( y = (y_i), y_i \in \{0, 1\}, i \in \mathbb{Z}^d \).

Along with the operator \( P_0 \) constructed above we introduce the operator \( P_0 = P_0 S_0(\varepsilon) \), which acts in the measure space on \( Y \) and has the form designated in the fundamental lemma. Here the Boolean function \( f_0 \) of \( r \) arguments is defined by the relation

\[
f_0(y_{V_1}, \ldots, y_{V_r}) = 0, \text{ if and only if } \begin{cases} y_{V_i} = 0 & \text{for all } V_i \text{ such that } L_k(V_i) < 0, \text{ or } \\ y_{V_i} = 0 & \text{for all } V_i \text{ such that } L_k(V_i) > 0, \text{ where } 1 \leq i \leq r. \end{cases} \tag{23}
\]

We also define the map \( H : X_m \to Y \) as \( H = h^2 \), where \( h : M_m \to M_1 \) is as follows: \( h(k) = 0, h(s) = 1, \) if \( s \neq k \). If \( \mu \) is a measure on \( X_m \), we denote by \( \mu H \) the measure induced on \( Y \) by \( \mu \) under the map \( H \). It now suffices to show that for any \( t \)

\[
P_0 H \prec HP_0^t \tag{24}
\]

in the sense of Mityushin [4]; note that (22) follows from the fundamental lemma and (24).

The operator \( P_0 \) is clearly monotone (since \( f_0 \) is monotone; see [4]). We therefore prove (24) by induction on the basis of the relation

\[
P_0 H \prec HP_0^t. \tag{25}
\]

It suffices to prove that \( \delta_0 P_0 H \prec \delta_0 H P_0 \) for any \( a \in X_m \), i.e.,

\[
\delta_0 P_0 S_{V_1} H \prec \delta_0 H P_0 S_{V_1}. \tag{26}
\]
Both measures in (26) are such that all $y_i$ in it are independent; consequently, it is sufficient to prove the inequality only for the projections of those measures onto the $i$-th coordinate:

$$
\delta_a P_{\xi_1} S_{\xi_2} H(y : y_i = 1) \leq \delta_a H P_{\xi_1} S_{\xi_2}^*(y : y_i = 1), \text{ i.e. }
(1 - \varepsilon m^{-1}) \delta_a P_{\xi}(x : x_i \neq k) + \varepsilon \delta_a U_{\xi}(x : x_i = k) \leq
\leq \delta_a H P_{\xi_1}^*(y : y_i = 1) + \varepsilon \delta_a H P_{\xi_1}^*(y : y_i = 0).
$$

Clearly, each of the measures $\delta_a P_{\xi_1}, \delta_a H P_{\xi_0}$ is concentrated in a certain state. If condition (21) is satisfied for $\sigma_{i_1} + \nu_1, \ldots, \sigma_{i_l} + \nu_l$, then both sides of (27) are equal to $\varepsilon$. Otherwise the left-hand side is either $(1 - \varepsilon m^{-1})$ or $\varepsilon$, and the right-hand side is equal to 1. In either case (27) is satisfied. This proves the theorem.

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