MONOTONIC BINARY CELLULAR AUTOMATA

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The operator $P$ with local interaction acts upon an ensemble of islands (finite subsets of a $d$-dimensional integer-valued lattice $\mathbb{Z}^d$). Criteria are derived that make it possible to predict a number of properties of its iterations $P^t$ as $t \to \infty$.

§1. FORMULATIONS

We denote by $X^d$ the set of all distributions of zeros and ones on a $d$-dimensional integer-valued lattice:

$$X^d = \{x\}, x = (x_i), i \in \mathbb{Z}^d, x_i \in \{0, 1\}.\tag{1}$$

Assume that we are given a nonempty finite list $U = \{u_1, \ldots, u_r\} \subset \mathbb{Z}^d$ and Boolean function $f(a_1, \ldots, a_r)$. A binary cellular automaton is an operator $P: X^d \to X^d$ specified as follows:

$$(P x)_t = f(x_{t+u_1}, \ldots, x_{t+u_r}).\tag{2}$$

From the applied standpoint, cellular (or mosaic) automata are of interest as biological and computational models (see [1, 2], which contain many other references). This article is in line with other studies that investigate the algorithmic possibilities of predicting the behavior of such automata (see, e.g., [3, 4]). All the results of this paper refer to the case in which

$$f(0, \ldots, 0) = 0,\tag{3}$$

i.e., $P$ carries the "all zero" state into itself. We denote by $I_0(x)$ the set of those points $i \in \mathbb{Z}^d$, where $x_i = a$. We will call $x$ an island (state with finite carrier) if $I_0(x)$ is finite.

Definition 1. Automaton $P$ washes out island $x$ if there exists a $t$ such that $P^t x$ is an "all zero" state. Automaton $P$ is a washout automaton if it washes out every island in $X^d$ on which it acts.

Binary cellular automaton $P$ will be called monotonic if the Boolean function $f(a_1, \ldots, a_r)$ that specifies it is monotonic, i.e.,

$$a_1 \leq a_1', \ldots, a_r \leq a_r' \Rightarrow f(a_1, \ldots, a_r) \leq f(a_1', \ldots, a_r').\tag{4}$$

All the results of this paper refer to binary monotonic cellular automata, which for brevity will be called $P$-type automata. The fundamental result, embodied in Proposition 1, permits us to determine whether any $P$-type automaton is a washout automaton. This result can also be applied in studying probabilistic automata with local interaction. If a $P$-type operator is a washout operator, we can expect that all iterations of its product by fairly weak random noise will maintain the probability that $x_i = 1$ at a low level. This assumption was partially proved in [5] for the case $m = 2$ in formula (9) of the paper.

In contradistinction to this paper, paper [6] proves that it is impossible to algorithmically predict certain properties of binary cellular operators, also associated with washout of islands.

Let us proceed to our formulations. In everything that follows a $P$-type operator will simply be called operator $P$. Formula (2) and conditions (3) and (4) are assumed everywhere. We incorporate lattice $\mathbb{Z}^d$ into a real $d$-dimensional space $\mathbb{R}^d$ with the same coordinate origin and direction of the axes.

Definition 2. We call a set $\mu \subset \mathbb{R}^d$ a zero set (for a given operator $P$) if $I_0(x) \cap \mu = \emptyset \Rightarrow \emptyset \in I_0(P x)$, where $\emptyset$ is the coordinate origin. We denote by $\sigma P$ the intersection of convex hulls of all null subsets $U$.

Remark. Since $U$ is finite, $\sigma P$ can be readily constructed.

Proposition 1. Operator P is a washout operator if and only if σ_P is empty.

Proposition 2. If σ_P is empty, then there exists a constant λ_P > 0, such that for any island x we have \( t < \lambda_P D(I_t(x)) \Rightarrow I_t(P^t(x)) = \varnothing \), where \( D(I_t(x)) \) is the diameter of \( I_t(x) \) in euclidean metric.

Proposition 3. Assume that \( \tau \) is a natural number, \( \xi \in \mathbb{Z}^d \); then

a) there exists an island \( x \) such that
\[
I_t(P^t x) \supset I_t(x) + \xi,
\]
if and only if \( -\xi/\tau \in \sigma_P \);

b) if \( \sigma_P = -\xi/\tau \), then there exists an island \( x \) such that
\[
I_t(P^t x) = I_t(x) + \xi.
\]

The definition of \( \sigma_P \) implies that if \( \sigma_P \) is nonempty then it contains a rational point, and therefore there exists \( x, \xi, \) and \( \tau \) for which (5) holds.

Proposition 4. Assume that \( \sigma_P \) is nonempty. Then for any \( R > 0 \) there exists an island \( x \) such that for all natural \( t \)
\[
I_t(P^t x) \supset \left( (-t \sigma_P) + \text{III}(R) \right) \cap \mathbb{Z}^d.
\]

Here and below \( (-t \sigma_P) \) denotes the image of \( \sigma_P \) under homothetic transformation with center \( \vec{0} \) and coefficient \( -t \), while the expression \( \text{III}(R) \) denotes a ball with center \( \vec{0} \) and radius \( R \).

Proposition 5. Assume that \( \sigma_P \) is nonempty. Then there exists a constant \( \mu_P \) such that for any island \( x \) and any natural \( t \)
\[
I_t(P^t x) \supset (-t \sigma_P) + \text{III}(\mu_P \max_{\nu \in \nu(x)} |\xi|).
\]

Proposition 6. Assume that \( \sigma_P \) consists of one point \( \xi \). Let \( \nu(x) \) be the number of points in
\[
\left[ \text{III}(\mu_P \max_{\nu \in \nu(x)}) + Q^d \right] \cap \mathbb{Z}^d,
\]
where \( Q^d \) is a cube defined by the following condition: all coordinates lie between 0 and 1. Then if all sets \( I_t(P^t(x)) \), \( 0 \leq t \leq 2^\nu(x) \), are nonempty, \( P \) does not wash out \( x \).

Proposition 6 shows that if \( \sigma_P \) consists of one point, then the problem of recognizing islands \( x \) that are washed out by a given operator \( P \) is algorithmically solvable (compare with [6]).

\section{Proofs}

Proposition 1 will be proved in conjunction with other proofs (it follows from Propositions 2 and 3a). Let us now prove Proposition 2.

\textbf{Lemma 1.} For any operator \( P \) the set \( \sigma_P \) can be represented as the intersection of a finite number of closed null half-spaces in \( \mathbb{R}^d \) (i.e., half-spaces that are null sets).

\textbf{Proof.} Assume that \( U' \) is a null subset of \( U \). Its convex hull is a polytope and can therefore be represented as the intersection of a finite number of half-spaces. All these half-spaces contain \( U' \) and are therefore null. Taking the intersection of such intersections with respect to all null subsets \( U \), we obtain the required representation of \( \sigma_P \).

We denote by \( \pi_1, \ldots, \pi_s \) closed null half-spaces whose intersection yields \( \sigma_P \). We associate with each \( \pi_k \) a linear functional \( L_k \) on \( \mathbb{R}^d \) with norm 1, which is nonnegative on \( \pi_k \) and only on it. By Theorem 21.3 in [7] (a version of Helly's theorem), among these there are \( m \) functionals (which we will call \( L_1, \ldots, L_m \)) such that
\[
\sum_{i=1}^m L_i(x) = 0,
\]
where \( \lambda_i \) and \( \varepsilon \) are positive constants and \( m \leq d + 1 \). Assume that \( I \) is a nonempty compactum in \( \mathbb{R}^d \). We write
\[
D_t(I) = \sum_{i=1}^m \max L_i(\xi) + \varepsilon.
\]
It is easy to show that
\[
D_t(I) \leq \left( \sum_{i=1}^m |L_i| \right) D_t(I),
\]
where $D(i)$ is the diameter of $I$ in euclidean metric. Since $\pi_i$ is null, for any $\xi \in \mathbb{Z}^d$ we have $((\pi_i + \xi) \cap \mathbb{Z}^d) = I_0(x) \Rightarrow \xi \in I_0(Px)$. From this, for any island $x$ we have $\max_{\xi \in \{P\}} \{\xi \} \leq \max_{\xi \in \{x\}} \{\xi \} + \tilde{I}_t(0)$, provided that $I_t(x)$ and $I_t(Px)$ are nonempty. Multiplying by $\lambda_k$ and summing, we obtain $D_I(I_t(Px)) \leq D_I(I_t(x)) - \varepsilon$. From this $I_t(P^t(x))$ is empty for all $t > t_p(x)$, where

$$t_p(x) = \frac{1}{\varepsilon} D_I(I_t(x)) \leq \frac{1}{\varepsilon} \sum_{k=1}^m |\lambda_k| D(I_t(x)).$$

Proposition 2 is thus proved.

Let us explain the dynamics of washout of islands geometrically. Assume that $I_1, \ldots, I_m$ satisfy (9), where no intrinsic subset of them satisfies an analogous condition. Then, as we can readily show, a nonempty set of the form

$$\{\xi : \forall i = 1, \ldots, m, \quad L_i(\xi) \leq l_i\},$$

being factorized with respect to the maximum subspace on which $\forall i, L_i = L_i(\emptyset)$, is a simplex or a point. If $I_t(x) \subset (10)$, then $I_t(Px)$ belongs to a set of the same form, only with different $l_i$ (which differs from the former ones by constants that are independent of $x$). In other words, the sides of the simplex shift by constant distances. What is most important is that the simplex becomes smaller. As a result of employing $P$ repeatedly, the simplex disappears over a time proportional to its original linear dimensions. Island $x$ is manifestly washed out in the process.

Let us consider the case in which $\sigma P$ is nonempty.

Set $M \subset \mathbb{R}^d$ will be called thick with respect to vector $V \neq 0$ if no straight line parallel to $V$ intersects $M$ at exactly one point.

**Lemma 2.** For any nonzero $V_1, \ldots, V_S$ there exists in $\mathbb{R}^d$ a $d$-dimensional polytope $M \subset \mathbb{R}^d$ that is thick with respect to all $V_1, \ldots, V_S$.

**Proof.** We add (if necessary) vectors $V_{S+1}, \ldots, V_{S'}$ such that system of vectors $V_1, \ldots, V_{S'}$ is complete in $\mathbb{R}^d$. We define $M$ by the formula

$$M = \left\{ \sum_{i=1}^{S'} C_i V_i : 0 \leq C_i \leq 1 \right\}.$$  \hspace{1cm} (11)

Obviously, $M$ is the desired polytope.

**Lemma 3.** Let $\xi \in \sigma P$. Assume that $d$-dimensional polytope $M$ is thick with respect to all nonzero vectors of the form $u_1 - \xi, \ldots, u_r - \xi$. Then there exists a $\rho_M > 0$ such that for all $\rho > \rho_M$ and all $\eta \in \mathbb{R}^d$ the condition $I_t(x) = (\rho M + \eta) \cap \mathbb{Z}^d$ defines an island $x$ such that for all natural $t$

$$I_t(P^t(x)) \supseteq (\rho M + \eta - \xi) \cap \mathbb{Z}^d.$$  \hspace{1cm} (12)

**Proof.** A ball with center in $M$ will be called unsuitable if it intersects with spaces $\alpha_1, \ldots, \alpha_m$ of polytope $M$ whose intersection is empty. From considerations of compactness, the minimum radius of all unsuitable balls can be achieved and is positive. We denote it by $R_0 > 0$. We denote by $Q_0$ the maximum edge length of a $d$-dimensional cube whose sides are parallel to the coordinate axes in $\mathbb{R}^d$ (and in $\mathbb{Z}^d$), belonging to $M$. Obviously, $Q_0 > 0$. We write $S_0 = \max_{1 \leq i \leq r} |u_i - \xi|$ and set

$$\rho_M = \max \left\{ S_0/R_0, 1/Q_0 \right\}$$

and we will now prove the assertion of the lemma. The definition of $Q_0$ readily yields that $(\rho M + \eta) \cap \mathbb{Z}^d$ is nonempty for all $\eta \in \mathbb{R}^d$, and this gives us the inequality in (12). Let us prove the inclusion in (12). It suffices to show that

$$I_t(P^t(x)) \supseteq (\rho M + \eta - \xi) \cap \mathbb{Z}^d.$$  \hspace{1cm} (14)

We take any point $(\rho M + \eta - \xi) \cap \mathbb{Z}^d$. We can assume that this point is $\bar{0}$. Let us show that $\bar{0} \in I_t(Px)$. All points $u_1, \ldots, u_r$ belong to the ball $\Pi(S_0) + \xi$. If $\Pi(S_0) + \xi \subset \rho M + \eta$, the assertion is obvious. Assume this is not the case. Then $\Pi(S_0) + \xi$ intersects some faces $\alpha_1, \ldots, \alpha_m$ of the polytope $\rho M + \eta$. By the definition of $R_0$, the intersection $\bigcap_{1 \leq i \leq r} \alpha_i$ is nonempty. Obviously, it contains at least one vertex $\omega$ of $\rho M + \eta$. Assume that $\alpha_1, \ldots, \alpha_m$ is a complete list of the faces of $\rho M + \eta$ containing $\omega$. We denote by $\nu + \omega$ the intersection of $m'$
closed half-spaces containing $\rho M + \eta$ and bounded by hyperplanes passing through $\alpha_1, \ldots, \alpha_m$. Obviously, $\nu + \xi$ is the translation of convex cone $\nu$. Since $\Pi(S_\parallel) + \xi \parallel does not intersect spaces of $\rho M + \eta$ other than $\alpha_1, \ldots, \alpha_m$, we have $\Pi(S_\parallel) + \xi \parallel (\nu + \omega) = \Pi(S_\parallel) + \xi \parallel (\rho M + \eta)$. Therefore, it suffices to show that $0 \in I_\parallel (P^n)$, where $x^n$ is defined by the condition $I_\parallel (x^n) = (\nu + \omega) \cap Z^d$. Since $0 \in \rho M + \eta - \xi$, we have $\xi \in \rho M + \eta$ and thus $\xi \in \nu + \omega$. Therefore, $\nu + \xi \subset \nu + \omega$. Consequently, in view of the fact that $f$ is monotonic, it suffices to show that $0 \in I_\parallel (P^n)$, where $x^n$ is given by the condition

$$I_\parallel (x^n) = (\nu + \xi) \cap Z^d. \tag{15}$$

Let us assume the contrary: $0 \in I_\parallel (P^n)$.

Since $\omega$ is a vertex of $\rho M + \eta$, we can pass through $\omega$ a support hyperplane $\gamma$ to $\rho M + \eta$, where $\gamma \cap \rho M + \eta = \omega$. We denote by $\gamma'$ an open half-space bounded by $\gamma$ and that does not intersect $\rho M + \eta$. We write $\gamma'' = \gamma' \cap (\rho M + \eta)$. Since $\gamma' \cap (\rho M + \eta) = \phi$, we have $\gamma' \cap (\nu + \omega) = \phi$ and hence $\nu'' \cap \nu + \xi = \phi$ as well. Therefore, $\nu'' \cap Z^d = I_\parallel (x^n)$. But since half-space $\gamma''$ does not contain $\xi$, it cannot be null. Therefore, if $0 \in I_\parallel (P^n)$, there must exist a point $u_i$, $1 \leq i \leq r$, which appears in $I_\parallel (x^n)$ but not in $\gamma''$. Point $u_i$ cannot belong to $\nu + \xi$ in view of (15). Then points $u_i$ and $\xi$ are different, and the line passing through them has only one common point $\xi$ with $\nu + \xi$. Then a line parallel to it and passing through $\omega$ has only one common point $\nu + \omega$, and hence with $\rho M + \eta$; but this is impossible, since $\rho M + \eta$ is thick with respect to $u_i - \xi$.

Lemma 3 has been proved. Lemmas 2 and 3 yield that if $\sigma \rho$ is nonempty, then $P$ is not a washout operator. Thus Proposition 1 has been fully proved. These lemmas also yield Proposition 3a in one direction: if $-\xi/\tau \notin \sigma \rho$, then there exists an island $x$ satisfying (5).

Let us prove Proposition 4. Let $\sigma \rho$ be nonempty. Obviously, $\sigma \rho$ is a polytope. Assume that $\sigma \rho$ is a convex hull of point $\xi_1, \ldots, \xi_m$. Using Lemma 2, we can construct a $d$-dimensional polytope $M$ that is thick with respect to all vectors of the form $u_i - \xi_j$, where $1 \leq i \leq r, 1 \leq j \leq m$. We denote $\rho_\xi$ by formula (13), with the only difference that now we have

$$S_\xi = \max |u_i - \xi_j|.$$ 

Let $\rho > \rho_\xi$. We define $x$ by the condition $I_\parallel (x) = \rho M \cap Z^d$. We further require that

$$\rho M \supset \Pi(R + dD(\sigma \rho)). \tag{16}$$

Proposition 4 follows from the fact that

$$I_\parallel (P^n) \supset \bigcup_{1 \leq i, \ldots, \xi \leq m} \left( \rho M - \sum_{s=1}^t \xi_k \right) \cap Z^d$$

$$\supset \bigcup_{1 \leq i, \ldots, \xi \leq m} \left( \Pi(R + dD(\sigma \rho)) + \sum_{s=1}^t \xi_k \right) \cap Z^d \supset \{ (-\tau \sigma \rho) + \Pi(R) \} \cap Z^d.$$ 

Here the first inclusion is a consequence of Lemma 3, the second of expression (16), and the third follows from the formula

$$\bigcup_{1 \leq i, \ldots, \xi \leq m} \left[ \Pi(dD(\sigma \rho)) - \sum_{s=1}^t \xi_k \right] \supset (-\tau \sigma \rho), \tag{17}$$

which we will now prove. Assume that a point belongs to $-\tau \sigma \rho$. Then it has the form $-\tau \xi_l$, where $\xi \in \sigma \rho$. By Carathéodory's theorem, there exists $l \leq n + 1$ points from among the $\xi_1, \ldots, \xi_m$ (which we denote by $\xi_1, \ldots, \xi_l$) such that

$$\eta = \sum_{i=1}^l c_i \xi_i, \quad c_i > 0, \quad \sum_{i=1}^l c_i = 1.$$ 

We define the numbers $k_1, \ldots, k_l$ in such a way that the number of equal $i$ from among them is $[ct_i], 1 \leq i \leq l - 1$; the number of equal $l$ is $t - \sum_{i=1}^{l-1} [ct_i]$. Then, as we can readily see, the distance between point $-\tau \xi_l$, $-\sum_{u=1}^l \xi_k u$ does not exceed $dD(\sigma \rho)$, QED. Proposition 4 has thus been proved.
Let us prove Proposition 5. From Lemma 1 the set $\sigma_P$ (and hence $-\sigma_P$) can be specified by a finite system of linear inequalities. Let

$$-\sigma_P = \{ \xi : \langle \xi, V_i \rangle \leq \alpha, i = 1, \ldots, m, \langle \xi, V_i \rangle \leq \alpha \},$$

where $|V_i| = 1$.

We will call the inequality $\langle \xi, V \rangle \leq \alpha$, where $|V| = 1$, a support inequality for $-\sigma_P$ if all points $-\sigma_P$ satisfy it and it becomes an equality for at least one of them. It is easy to show that all support inequalities for $-\sigma_P$ can be represented as linear combinations of inequalities

$$\langle \xi, V_i \rangle \leq \alpha, \quad i = 1, \ldots, m,$$

with nonnegative coefficients and uniformly bounded sums of these coefficients. We denote the constant that bounds them by $\mu_P$ and prove Proposition 5 for this $\mu_P$. We write $\Sigma_\beta = \{ \xi : \langle \xi, V_i \rangle \leq \alpha + \beta, i = 1, \ldots, m \}$. Obviously, $\Sigma_0 = -\sigma_P$.

First we prove that for all $\beta \geq 0$

$$\Sigma_\beta = \Sigma_\alpha + \III (\mu_P \beta).$$

(19)

Let $\eta_1 \in \Sigma_\beta$, $\eta_1 \notin \Sigma_0$, and let $\eta_0$ be the point in $\Sigma_0$ that is closest to $\eta_1$. Then $\eta_1 \in \{ \xi : \langle \xi, \eta_1 - \eta_0 \rangle \leq \langle \eta_0, \eta_1 - \eta_0 \rangle \}$. We represent the support inequality for $\Sigma_0$

$$\langle \xi, \eta_1 - \eta_0 \rangle \leq \langle \eta_0, \eta_1 - \eta_0 \rangle$$

as the sum of inequalities (18) with coefficients $k_i \geq 0$, where $\sum_{i=1}^m k_i \leq \mu_P$. Then for all $\xi \in \Sigma_\beta$

$$\sum_{i=1}^m k_i \langle \xi, V_i \rangle \leq \sum_{i=1}^m k_i (\alpha_i + \beta),$$

from which

$$\langle \xi, \eta_1 - \eta_0 \rangle \leq \langle \eta_0, \eta_1 - \eta_0 \rangle + \mu_P \beta.$$

Replacing $\xi$ by $\eta_1$, we obtain $|\eta_1 - \eta_0| \leq \mu_P \beta$. Condition (19) has been proved. Now we note that for any $i = 1, \ldots, m, \xi \in I_i(\kappa) \Rightarrow \langle \xi, V_i \rangle \max |\xi|$. It is easy to see that then for any $i = 1, \ldots, m, \xi \in I_i(P_*^{\kappa}) \Rightarrow \langle \xi, V_i \rangle \leq t \omega + \max |\xi|$. Thus, $t^{-1} I_i(P_*^{\kappa}) \subset \Sigma_0$. Then expression (17) yields $t^{-1} I_i(P_*^{\kappa}) = (-\sigma_P) + \III (t^{-1} \mu_P \max |\xi|)$. Multiplying this by $t$, we obtain (6). Proposition 5 has been proved. It yields the missing part of Proposition 3a. Proposition 3b can be readily derived from Propositions 3a and 5. Proposition 6 can be readily proved by using Proposition 5.

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**LITERATURE CITED**