MONOTONIC BINARY CELLULAR AUTOMATA

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The operator P with local interaction acts upon an ensemble of islands (finite subsets of a d-dimensional integer-valued lattice Z^d). Criteria are derived that make it possible to predict a number of properties of its iterations P^t as $t \to \infty$.

§1. FORMULATIONS

We denote by X^d the set of all distributions of zeros and ones on a d-dimensional integer-valued lattice:

$$X^d=\{x\},\ x=(x_\xi),\ \xi\in Z^d,\ x_\xi\in\{0, 1\}. \quad (1)$$

Assume that we are given a nonempty finite list $U = \{u_1, \ldots, u_r\} \subset Z^d$ and Boolean function $f(a_1, \ldots, a_r)$. A binary cellular automaton is an operator $P: X^d \to X^d$ specified as follows:

$$(Px)_\xi=f(x_{\xi+u_1}, \ldots, x_{\xi+u_r}). \quad (2)$$

From the applied standpoint, cellular (or mosaic) automata are of interest as biological and computational models (see [1, 2], which contain many other references). This article is in line with other studies that investigate the algorithmic possibilities of predicting the behavior of such automata (see, e.g., [3, 4]). All the results of this paper refer to the case in which

$$f(0, \ldots, 0)=0, \quad (3)$$

i.e., P carries the "all zero" state into itself. We denote by $I_a(x)$ the set of those points $\xi \in Z^d$, where $x_\xi = a$. We will call x an island (state with finite carrier) if $I_1(x)$ is finite.

Definition 1. Automaton P washes out island x if there exists a t such that $P^t x$ is an "all zero" state. Automaton P is a washout automaton if it washes out every island in X^d on which it acts.

Binary cellular automaton P will be called monotonic if the Boolean function $f(a_1, \ldots, a_r)$ that specifies it is monotonic, i.e.,

$$a_1\leqslant a_1', \ldots, a_r\leqslant a_r' \Rightarrow f(a_1, \ldots, a_r)\leqslant f(a_1', \ldots, a_r'). \quad (4)$$

All the results of this paper refer to binary monotonic cellular automata, which for brevity will be called P-type automata. The fundamental result, embodied in Proposition 1, permits us to determine whether any P-type automaton is a washout automaton. This result can also be applied in studying probabilistic automata with local interaction. If a P-type operator is a washout operator, we can expect that all iterations of its product by fairly weak random noise will maintain the probability that $x_\xi = 1$ at a low level. This assumption was partially proved in [5] [for the case m = 2 in formula (9) of the paper].

In contradistinction to this paper, paper [6] proves that it is impossible to algorithmically predict certain properties of binary cellular operators, also associated with washout of islands.

Let us proceed to our formulations. In everything that follows a P-type operator will simply be called operator P. Formula (2) and conditions (3) and (4) are assumed everywhere. We incorporate lattice Z^d into a real d-dimensional space R^d with the same coordinate origin and direction of the axes.

Definition 2. We call a set $\mu \subset R^d$ a zero set (for a given operator P) if $I_1(x) \cap \mu = \phi \Rightarrow \bar{0} \in I_0(Px)$, where $\bar{0}$ is the coordinate origin. We denote by σ_P the intersection of convex hulls of all null subsets U.

Remark. Since U is finite, σ_P can be readily constructed.

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Proposition 1. Operator P is a washout operator if and only if σ_P is empty.

Proposition 2. If σ_P is empty, then there exists a constant $\lambda_P > 0$, such that for any island x we have $t < \lambda_P D(I_1(x)) \Rightarrow I_1(P^t(x)) = \phi$, where $D(I_1(x))$ is the diameter of $I_1(x)$ in euclidean metric.

Proposition 3. Assume that τ is a natural number, $\xi \in Z^d$; then

a) there exists an island x such that

$$I_1(P^\tau x) \supset I_1(x) + \xi, \tag{5}$$

if and only if $-\xi/\tau \in \sigma_P$;

b) if $\sigma_P = -\xi/\tau$, then there exists an island x such that

$$I_1(P^\tau x) = I_1(x) + \xi. \tag{6}$$

The definition of σ_P implies that if σ_P is nonempty then it contains a rational point, and therefore there exists x, ξ, and τ for which (5) holds.

Proposition 4. Assume that σ_P is nonempty. Then for any $R > 0$ there exists an island x such that for all natural t

$$I_1(P^t x) \supset [(-t\sigma_P) + \text{III}(R)] \cap Z^d. \tag{7}$$

Here and below $(-t\sigma_P)$ denotes the image of σ_P under homothetic transformation with center $\bar{0}$ and coefficient $-t$, while the expression III(R) denotes a ball with center $\bar{0}$ and radius R.

Proposition 5. Assume that σ_P is nonempty. Then there exists a constant μ_P such that for any island x and any natural t

$$I_1(P^t x) \subset (-t\sigma_P) + \text{III}(\mu_P \max_{\xi \in I_1(x)} |\xi|). \tag{8}$$

Proposition 6. Assume that σ_P consists of one point ξ. Let $\nu(x)$ be the number of points in $[\text{III}(\mu_P \max_{\eta \in I_1(x)} |\eta|) + Q^d] \cap Z^d$, where Q^d is a cube defined by the following condition: all coordinates lie between 0 and 1. Then if all sets $I_1(P^t(x)), 0 \le t \le 2^{\nu(x)}$, are nonempty, P does not wash out x.

Proposition 6 shows that if σ_P consists of one point, then the problem of recognizing islands x that are washed out by a given operator P is algorithmically solvable (compare with [6]).

§2. PROOFS

Proposition 1 will be proved in conjunction with other proofs (it follows from Propositions 2 and 3a). Let us now prove Proposition 2.

LEMMA 1. For any operator P the set σ_P can be represented as the intersection of a finite number of closed null half-spaces in R^d (i.e., half-spaces that are null sets).

Proof. Assume that U' is a null subset of U. Its convex hull is a polytope and can therefore be represented as the intersection of a finite number of half-spaces. All these half-spaces contain U' and are therefore null. Taking the intersection of such intersections with respect to all null subsets U, we obtain the required representation of σ_P.

We denote by $\pi_1, \ldots, \pi_s$ closed null half-spaces whose intersection yields σ_P. We associate with each π_k a linear functional L_k on R^d with norm 1, which is nonegative on π_k and only on it. By Theorem 21.3 in [7] (a version of Helly's theorem), among these there are m functionals (which we will call $L_1, \ldots, L_m$) such that

$$\sum_{i=1}^{m} \lambda_i L_i + \varepsilon \equiv 0, \tag{9}$$

where λ_i and ε are positive constants and $m \le d+1$. Assume that I is a nonempty compactum in R^d. We write

$$D_1(I) = \sum_{i=1}^{m} \max L_i(\xi) + \varepsilon.$$

It is easy to show that

$$D_1(I) \leqslant \left(\sum_{i=2}^{m} |\lambda_i|\right) D(I),$$

where $D(I)$ is the diameter of I in euclidean metric. Since π_i is null, for any $\xi \in Z^d$ we have $[(\pi_i + \xi) \cap Z^d] \subset I_0(x) \Rightarrow \xi \in I_0(Px)$. From this, for any island x we have $\max_{\xi \in I_1(Px)} L_i(\xi) \leq \max_{\xi \in I_1(x)} L_i(\xi) + L_i(\bar{0})$, provided that $I_1(x)$ and $I_1(Px)$ are nonempty. Multiplying by λ_k and summing, we obtain $D_1(I_1(Px)) \leq D_1(I_1(x)) - \varepsilon$. From this $I_1(P^t x)$ is empty for all $t > t_P(x)$, where

$$t_P(x) \leqslant \frac{1}{\varepsilon} D_1(I_1(x)) \leqslant \frac{1}{\varepsilon} \sum_{k=2}^{m} |\lambda_k| D(I_1(x)).$$

Proposition 2 is thus proved.

Let us explain the dynamics of washout of islands geometrically. Assume that $L_1, \ldots, L_m$ satisfy (9), where no intrinsic subset of them satisfies an analogous condition. Then, as we can readily show, a nonempty set of the form

$$\{\xi : \forall i=1, \ldots, m, \quad L_i(\xi) \leqslant l_i\}, \tag{10}$$

being factorized with respect to the maximum subspace on which $\forall i$, $L_i \equiv L_i(\bar{0})$, is a simplex or a point. If $I_1(x) \subset (10)$, then $I_1(Px)$ belongs to a set of the same form, only with difference l_i (which differs from the former ones by constants that are independent of x). In other words, the sides of the simplex shift by constant distances. What is most important is that the simplex becomes smaller. As a result of employing P repeatedly, the simplex disappears over a time proportional to its original linear dimensions. Island x is manifestly washed out in the process.

Let us consider the case in which σ_P is nonempty.

Set $M \subset R^d$ will be called thick with respect to vector $V \neq \bar{0}$ if no straight line parallel to V intersects M at exactly one point.

LEMMA 2. For any nonzero $V_1, \ldots, V_s$ there exists in R^d a d-dimensional polytope $M \subset R^d$ that is thick with respect to all $V_1, \ldots, V_s$.

Proof. We add (if necessary) vectors $V_{s+1}, \ldots, V_{s'}$ such that system of vectors $V_1, \ldots, V_{s'}$ is complete in R^d. We define M by the formula

$$M = \left\{ \sum_{i=1}^{s'} C_i V_i, \quad 0 \leqslant C_i \leqslant 1 \right\}. \tag{11}$$

Obviously, M is the desired polytope.

LEMMA 3. Let $\xi \in \sigma_P$. Assume that d-dimensional polytope M is thick with respect to all nonzero vectors of the form $u_1 - \xi, \ldots, u_r - \xi$. Then there exists a $\rho_M > 0$ such that for all $\rho > \rho_M$ and all $\eta \in R^d$ the condition $I_1(x) = (\rho M + \eta) \cap Z^d$ defines an island x such that for all natural t

$$I_1(P^t x) \supset (\rho M + \eta - t\xi) \cap Z^d \neq \emptyset. \tag{12}$$

Proof. A ball with center in M will be called unsuitable if it intersects with spaces $\alpha_1, \ldots, \alpha_m$ of polytope M whose intersection is empty. From considerations of compactness, the minimum radius of all unsuitable balls can be achieved and is positive. We denote it by $R_0 > 0$. We denote by Q_0 the maximum edge length of a d-dimensional cube whose sides are parallel to the coordinate axes in R^d (and in Z^d), belonging to M. Obviously, $Q_0 > 0$. We write $S_0 = \max_{1 \leqslant i \leqslant r} |u_i - \xi|$ and set

$$\rho_M = \max \{S_0/R_0, 1/Q_0\} \tag{13}$$

and we will now prove the assertion of the lemma. The definition of Q_0 readily yields that $(\rho M + \theta) \cap Z^d$ is nonempty for all $\theta \in R^d$, and this gives us the inequality in (12). Let us prove the inclusion in (12). It suffices to show that

$$I_1(Px) \supset (\rho M + \eta - \xi) \cap Z^d. \tag{14}$$

We take any point $(\rho M + \eta - \xi) \cap Z^d$. We can assume that this point is $\bar{0}$. Let us show that $\bar{0} \in I_1(Px)$. All points $u_1, \ldots, u_r$ belong to the ball $III(S_0) + \xi$. If $III(S_0) + \xi \subset \rho M + \eta$, the assertion is obvious. Assume this is not the case. Then $III(S_0) + \xi$ intersects some faces $\alpha_1, \ldots, \alpha_m$ of the polytope $\rho M + \eta$. By the definition of R_0, the intersection $\bigcap_{i=1}^{m} \alpha_i$ is nonempty. Obviously, it contains at least one vertex ω of $\rho M + \eta$. Assume that $\alpha_1, \ldots, \alpha_{m'}$ is a complete list of the faces of $\rho M + \eta$ containing ω. We denote by $\nu + \omega$ the intersection of m' closed half-spaces containing $\rho M + \eta$ and bounded by hyperplanes passing through $\alpha_1, \ldots, \alpha_{m'}$. Obviously, $\nu + \omega$ is the translation of convex cone ν. Since $III(S_0) + \xi$ does not intersect spaces of $\rho M + \eta$ other than $\alpha_1, \ldots, \alpha_m$, we have $[III(S_0) + \xi] \cap (\nu + \omega) = [III(S_0) + \xi] \cap (\rho M + \eta)$. Therefore, it suffices to show that $\bar{0} \in I_1(Px')$, where x' is defined by the condition $I_1(x') = (\nu + \omega) \cap Z^d$. Since $\bar{0} \in \rho M + \eta - \xi$, we have $\xi \in \rho M + \eta$ and thus $\xi \in \nu + \omega$. Therefore, $\nu + \xi \subset \nu + \omega$. Consequently, in view of the fact that f is monotonic, it suffices to show that $\bar{0} \in I_1(Px'')$, where x'' is given by the condition

$$I_1(x'') = (\nu + \xi) \cap Z^d. \tag{15}$$

Let us assume the contrary: $\bar{0} \in I_0(Px'')$.

Since ω is a vertex of $\rho M + \eta$, we can pass through ω a support hyperplane γ to $\rho M + \eta$, where $\gamma \cap \rho M + \eta = \omega$. We denote by γ' an open half-space bounded by γ and that does not intersect $\rho M + \eta$. We write $\gamma'' = \gamma' + \xi - \omega$. Since $\gamma' \cap (\rho M + \eta) = \phi$, we have $\gamma' \cap \nu + \omega = \phi$, and hence $\gamma'' \cap \nu + \xi = \phi$ as well. Therefore, $\gamma'' \cap Z^a \subset I_0(x'')$. But since half-space γ'' does not contain ξ, it cannot be null. Therefore, if $\bar{0} \in I_0(Px')$, there must exist a point u_i, $1 \le i \le r$, which appears in $I_0(x'')$ but not in γ''. Point u_i cannot belong to $\nu + \xi$ in view of (15). Then points u_i and ξ are different, and the line passing through them has only one common point ξ with $\nu + \xi$. Then a line parallel to it and passing through ω has only one common point ω with $\nu + \omega$, and hence with $\rho M + \eta$; but this is impossible, since $\rho M + \eta$ is thick with respect to $u_i - \xi$.

Lemma 3 has been proved. Lemmas 2 and 3 yield that if σ_P is not empty, then P is not a washout operator. Thus Proposition 1 has been fully proved. These lemmas also yield Proposition 3a in one direction: if $-\xi/\tau \in \sigma_P$, then there exists an island x satisfying (5).

Let us prove Proposition 4. Let σ_P be nonempty. Obviously, σ_P is a polytope. Assume that σ_P is a convex hull of point $\xi_1, \ldots, \xi_m$. Using Lemma 2, we can construct a d-dimensional polytope M that is thick with respect to all vectors of the form $u_i - \xi_j$, where $1 \le i \le r$, $1 \le j \le m$. We define ρ_0 by formula (13), with the only difference that now we have

$$S_0 = \max_{\substack{1 \le i \le r \\ 1 \le j \le m}} |u_i - \xi_j|.$$

Let $\rho > \rho_0$. We define x by the condition $I_1(x) = \rho M \cap Z^d$. We further require that

$$\rho M \supset III(R + dD(\sigma_P)). \tag{16}$$

Proposition 4 follows from the fact that

$$I_1(P^t x) \supset \Big[\bigcup_{1 \le k_1, \ldots, k_t \le m} \Big(\rho M - \sum_{u=1}^{t} \xi_{k_u}\Big)\Big] \cap Z^d$$

$$\supset \Big\{ \bigcup_{1 \le k_1, \ldots, k_t \le m} \Big[III(R + dD(\sigma_P)) + \sum_{u=1}^{t} \xi_{k_u}\Big]\Big\} \cap Z^d \supset [(-t\sigma_P) + III(R)] \cap Z^d.$$

Here the first inclusion is a consequence of Lemma 3, the second of expression (16), and the third follows from the formula

$$\bigcup_{1 \le k_1, \ldots, k_t \le m} \Big[III(dD(\sigma_P)) - \sum_{u=1}^{t} \xi_{k_u} \Big] \supset (-t\sigma_P), \tag{17}$$

which we will now prove. Assume that a point belongs to $-t\sigma_P$. Then it has the form $-t\eta$, where $\eta \in \sigma_P$. By Caratheodory's theorem, there exists $l \le n + 1$ points from among the $\xi_1, \ldots, \xi_m$ (which we denote by $\xi_1, \ldots, \xi_l$) such that

$$\eta = \sum_{i=1}^{l} c_i \xi_i, \quad c_i > 0. \quad \sum_{i=1}^{l} c_i = 1.$$

We define the numbers $k_1, \ldots, k_t$ in such a way that the number of equal i from among them is $[c_i t]$, $1 \le i \le l - 1$; the number of equal l is $t - \sum_{i=1}^{l-1} [c_i t]$. Then, as we can readily see, the distance between point $-t\eta$, $-\sum_{u=1}^{t} \xi_{k_u}$ does not exceed $dD(\sigma_P)$. QED. Proposition 4 has thus been proved.

Let us prove Proposition 5. From Lemma 1 the set σ_P (and hence $-\sigma_P$) can be specified by a finite system of linear inequalities. Let

$$-\sigma_P=\{\xi : \forall i=1,\ldots,m,\ \langle \xi, V_i\rangle \leqslant \alpha_i\}, \text{where } |V_i|=1.$$

We will call the inequality $\langle \xi, V\rangle \leq \alpha$, where $|V| = 1$, a support inequality for $-\sigma_P$ if all points $-\sigma_P$ satisfy it and it becomes an equality for at least one of them. It is easy to show that all support inequalities for $-\sigma_P$ can be represented as linear combinations of inequalities

$$\langle \xi, V_i\rangle \leqslant \alpha_i, \qquad i=1,\ldots,m, \tag{18}$$

with nonnegative coefficients and uniformly bounded sums of these coefficients. We denote the constant that bounds them by μ_P and prove Proposition 5 for this μ_P. We write $\Sigma_\beta = \{\xi : \forall i = 1, \ldots, m,\ \langle \xi, V_i\rangle \leq \alpha_i + \beta\}$. Obviously, $\Sigma_0 = -\sigma_P$.

First we prove that for all $\beta \geq 0$

$$\Sigma_\beta \subset \Sigma_0 + \text{III}(\mu_P\beta). \tag{19}$$

Let $\eta_1 \in \Sigma_\beta$, $\eta_1 \notin \Sigma_0$, and let η_0 be the point in Σ_0 that is closest to η_1. Then $\Sigma_0 \subset \{\xi : \langle \xi, \eta_1 - \eta_0\rangle \leq \langle \eta_0, \eta_1 - \eta_0\rangle\}$. We represent the support inequality for Σ_0

$$\left\langle \xi, \frac{\eta_1-\eta_0}{|\eta_1-\eta_0|}\right\rangle \leqslant \left\langle \eta_0; \frac{\eta_1-\eta_0}{|\eta_1-\eta_0|}\right\rangle$$

as the sum of inequalities (18) with coefficients $k_i \geq 0$, where $\sum_{i=1}^{m} k_i \leq \mu_P$. Then for all $\xi \in \Sigma_\beta$

$$\sum_{i=1}^{m} k_i\langle \xi, V_i\rangle \leqslant \sum_{i=1}^{m} k_i(\alpha_i+\beta),$$

from which

$$\left\langle \xi, \frac{\eta_1-\eta_0}{|\eta_1-\eta_0|}\right\rangle \leqslant \left\langle \eta_0, \frac{\eta_1-\eta_0}{|\eta_1-\eta_0|}\right\rangle + \mu_P\beta.$$

Replacing ξ by η_1, we obtain $|\eta_1 - \eta_0| \leq \mu_P\beta$. Condition (19) has been proved. Now we note that for any $i = 1, \ldots, m$, $\xi \in I_1(x) \Rightarrow \langle \xi, V_i\rangle \max_{\xi \in I_1(x)} |\xi|$. It is easy to see that then for any $i = 1, \ldots, m$, $\xi \in I_1(P^t x) \Rightarrow \langle \xi, V_i\rangle \leq t\alpha_i + \max_{\xi \in I_1(x)} |\xi|$. Thus, $t^{-1} I_1(P^t x) \subset \Sigma t^{-1} \max_{\xi \in I_1(x)} |\xi|$. Then expression (17) yields $t^{-1} I_1(P^t x) \subset (-\sigma_P) + \text{III}(t^{-1}\mu_P \max_{\xi \in I_1(x)} |\xi|$. Multiplying this by t, we obtain (8). Proposition 5 has been proved. It yields the missing part of Proposition 3a. Proposition 3b can be readily derived from Propositions 3a and 5. Proposition 6 can be readily proved by using Proposition 5.

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