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A non-linear eroder in presence of one-sided noise

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Abstract: We study a class of cellular automata, that is random operators acting on normed measures on the space $\{0, \ldots, m\}^{\mathbb{Z}^d}$ which can be presented as superpositions $F_r D$, where D is a monotonic deterministc operator with uniform local interaction and F_r turns every component into the maximal state m with probability r independently from fate of other components. We call an *island* any configuration, whose set of components with non-zero state is finite, but not empty. We assume that D transforms the configuration "all zeros" into itself and say that D *erodes* an island x if there is t such that $D^t x =$"all zeros". We say that D is an *eroder* if it erodes all islands. We say that D is a *linear eroder* if D erodes any island in a time which does not exceed a linear function of diameter of this island. Two special cases have been studied before: one with $m = 1$ and another with $d = 1$. In both cases necessary and sufficient conditions for an eroder have been presented and all eroders are linear. We find that as soon as $m > 1$ and $d > 1$, there are non-linear eroders. We concentrate our attention on one cellular automaton G with $m = d = 2$ and show that $F_r G$ is ergodic for all $r > 0$.

Key words: Cellular automata, critical droplet, ergodicity, eroder, metastability.

1 Introduction

It is common to believe that studies of ergodicity of cellular automata or CA for short are relevant to statistical physics. However, it is well-known that for too large classes of CA their ergodicity is undecidable (Kurdyumov, 1978 and Toom, 2000a,b). So, to obtain positive results, one has to reduce attention to some special class of CA.

In our case the space is $\mathbb{Z}^d$ and all its elements are called *points*. The configuration space is $\Omega = \{0, \ldots, m\}^{\mathbb{Z}^d}$ and its elements are called *configurations*. Any configuration x has *components* $x_v \in \{0, \ldots, m\}$ at all points $v \in \mathbb{Z}^d$. We denote $\mathcal{M}$ the set of normed measures on Ω (that is, on the σ-algebra generated by cylinder sets). As usual, a measure $\mu \in \mathcal{M}$ is called *invariant* for an operator $P : \mathcal{M} \to \mathcal{M}$ if $P\mu = \mu$ and P is called *ergodic* if it has exactly one invariant measure μ_{inv} and

$$\forall\ \mu \in \mathcal{M} : \lim_{t\to\infty} P^t\mu = \mu_{inv},$$

where convergence means convergence on all cylinder sets.

In fact, we consider superpositions $F_r D$ (first D, then F_r) acting on $\mathcal{M}$. Here F_r is the well-known random operator, which turns the state of every component into m with probability r independently from fate of other components. D is a uniform monotonic deterministic operator with local interaction. To describe a specific D, we choose a non-empty finite list of vectors $v_1, \ldots, v_n$ and a function

$$f : \{0, \ldots, m\}^n \to \{0, \ldots, m\}$$

and define the value of v-th component after application of D to any configuration x as follows:

$$(Dx)_v = f(x_{v+v_1}, \ldots, x_{v+v_n}). \tag{1.1}$$

We denote by τ_v the shift of the space $\mathbb{Z}^d$ at a vector v and use the same notation for the corresponding shifts of Ω and $\mathcal{M}$. We call an operator from Ω to Ω or from $\mathcal{M}$ to $\mathcal{M}$ *uniform* if it commutates with all shifts. It is evident that F_r and D are uniform.

However, this approach still leads to undecidabilities, as was shown by Petri (1987), so we additionally assume monotonicity of D, which means the following. Given two configurations $x, y \in \Omega$, we say that $x \prec y$ or $y \succ x$ if $x_v \le y_v$ for all $v \in \mathbb{Z}^d$. We call D monotonic if

$$x \prec y \Longrightarrow Dx \prec Dy.$$

This amounts to saying that f used in (1.1) is monotonic, that is

$$x_1 \le y_1, \ldots, x_n \le y_n \Longrightarrow f(x_1, \ldots, x_n) \le f(y_1, \ldots, y_n).$$

Since monotonicity is assumed, it is not a great loss of generality to assume also that

$$f(0, \ldots, 0) = 0 \quad \text{and} \quad f(m, \ldots, m) = m. \tag{1.2}$$

Now to define monotonicity of stochastic operators. A real function F on Ω is called monotonic if

$$x \prec y \Longrightarrow F(x) \le F(y).$$

Given two measures μ and ν on our configuration space, we say that $\mu \prec \nu$ or $\nu \succ \mu$ if for every monotonic function F, we have

$$E(F \mid \mu) \le E(F \mid \nu),$$

where $E(\cdot \mid \cdot)$ is expectation. We shall say that a sequence μ_n is non-decreasing if $\mu_n \prec \mu_{n+1}$ for all n. An operator P on $\mathcal{M}$ is called monotonic if

$$\mu \prec \nu \Longrightarrow P\mu \prec P\nu.$$

Any measure concentrated in one configuration is called a δ-measure. We shall always denote such measures using the letter δ with various indices. In particular, δ_a means the measure concentrated in the configuration "all a", all of whose components equal a.

Due to (1.2), the measure δ_m concentrated in the configuration "all m" is invariant for D. Of course, δ_m is invariant for F_r also, so it is invariant for their superposition $F_r D$. Therefore ergodicity of $F_r D$ is equivalent to

$$\forall\ \mu \in \mathcal{M} : \lim_{t\to\infty} P^t \mu = \delta_m.$$

From monotonicity, this condition is equivalent to

$$\lim_{t\to\infty} P^t \delta_0 = \delta_m,$$

where δ_0 is the measure concentrated in the configuration "all zeros".

Since the operator F_r is quite simple and standard, all the originally is concentrated in D. We want to know, which properties of D are important for ergodicity of $F_r D$. We call a configuration x an *island* if the set $\{v \in \mathbb{Z}^d : x_v \neq 0\}$ is finite, but not empty. (So "all zeros" is not an island.) Due to (1.2), $f(0,\ldots,0) = 0$. Thus D transforms every island into an island or into "all zeros".

We call *toptime* of an island x and denote $\text{toptime}_D(x)$ the smallest t such that all components of $D^t x$ are less than m. If there is no such t, the toptime is infinite by definition. Our main results are presented below as Theorems 1 and 2. To provide a background, we start with a simple Theorem 0.

Theorem 0. *Given D, suppose that there is an island, whose toptime is infinite. Then $F_r D$ is ergodic for all $r > 0$.*

Let us prove Theorem 0 for the reader's convenience and to prepare proofs of our main Theorems 1 and 2. Throughout this article we shall present our process as a measure on the space

$$\overline{\Omega} = \{0, 1, \ldots, m\}^{\mathbb{Z}^d \cdot \mathbb{Z}_+}$$

induced by a product-measure $\mathcal{P}$ on the auxiliary space

$$\Gamma = \{0, 1\}^{\mathbb{Z}^d \cdot \mathbb{Z}_+}.$$

We denote $\gamma \in \Gamma$ the elements of Γ and γ_v^t the components of γ indexed by $v \in \mathbb{Z}^d$ and $t \in \mathbb{Z}_+$.

Let $\mathcal{P}$ be a product-measure on Γ such that

$$\gamma_v^t = \begin{cases} 1 & \text{with probability} \quad \text{r}, \\ 0 & \text{with probability} \quad \text{1-r}. \end{cases}$$

Our process is induced by $\mathcal{P}$ with a map defined in the following inductive way. Base of induction: $x_v^0 = 0$ for all $v \in \mathbb{Z}^d$, where x_v^t are components of configuration x in $\overline{\Omega}$. Induction step for all $t = 0, 1, 2, \ldots$:

$$x_v^{t+1} = \begin{cases} m & \text{if } \ \gamma_v^{t+1} = 1, \\ f(x_{v+v_1}^t, \ldots, x_{v+v_n}^t) & \text{otherwise.} \end{cases}$$

We shall use the letter $\mathcal{P}$ also to denote probabilities of events of our process meaning probabilities of their pre-images in Γ. Due to uniformity, Theorem 0 is equivalent to

$$\lim_{T\to\infty} P(x_0^T = m) = 1. \tag{1.3}$$

Let us denote z the island, whose toptime is infinite. Then for every natural t there is v_t such that $(D^t z)_{v_t} = m$. Therefore, due to uniformity,

$$(D^t(\tau_{-v_t} z))_0 = m.$$

For each $t \in [1, T]$, we consider the event

$$E_t \;=\; \{\gamma : \gamma_w^t = 1 \text{ for all } w \text{ such that } (\tau_{-v_{T-t}} z)_w \neq 0\}.$$

Due to monotonicity, this event is a guarantee that the configuration at time t is not less than $\tau_{-v_{T-t}} z$, which, in its turn, is a guarantee that $x_0^T = m$. We know that $\gamma_v^t = 1$ with probability $r > 0$ independently for each pair (v, t). Therefore $\mathcal{P}(E_t) = r^k$, where k is the number of non-zero components of z. Thus $\mathcal{P}(\text{ not } E_t) = 1 - r^k$. Since events E_t are independent from each other,

$$\mathcal{P}\left(\text{not } E_t \text{ for all } t \in [1, T]\right) = (1 - r^k)^T.$$

Therefore

$$\mathcal{P}(x_0^T = m) \geq 1 - (1 - r^k)^T.$$

The right side of this inequality tends to 1 when $T \to \infty$, which implies (1.3). *Theorem 0 is proved.*

Our main Theorems 1 and 2 pertain to the line of study concerned with ergodicity of superpositions $F_r D$. It is well-known that every such superposition is ergodic for r close enough to 1 . However, for small values of $r > 0$ these superpositions behave in different ways. Some attempts have been made to find properties of D relevant to existence of $r > 0$ for which $F_r D$ is non-ergodic. In this connection it is common to say that D *erodes* an island x if there is t such that $D^t x =$"all zeros". The smallest t with this property is denoted $\text{lifetime}_D(x)$ and called the *lifetime* of this island. If there is no such t, lifetime of x equals ∞ by definition. D is called an *eroder* if it erodes all islands. Toom (1980) examined the case $m = 1$ and arbitrary d, gave a condition for D to be an eroder and proved that D is an eroder if and only if $F_r D$ is not ergodic for small enough $r > 0$. Galperin (1976) examined the case $d = 1$ and arbitrary m and gave a condition for D to be an eroder in this case. However, Toom (1976) showed that in this case there is an eroder D such that $F_r D$ is ergodic for all $r > 0$. All these results were obtained under the assumption of monotonicity of D. Without this assumption, as Petri (1987) showed, both the set of eroders and the set of non-eroders are non-enumerable even in the case $d = 1$.

For any island x, we call its *diameter* $\text{diam}(x)$ the greatest Euclidean distance between v and w such that $x_v \neq 0$ and $x_w \neq 0$. Thus, every island has a finite diameter. (Remember that "all zeros" is not an island.) An eroder D is called *linear* if there is a number c such that the lifetime of any island x does not exceed $c(\mathrm{diam}(x)+1)$. Toom (1979) and Galperin (1976) showed that in all the cases considered by them, namely $m=1$ or $d=1$, all eroders are linear. As a contrast with this, we shall show that as soon as $m \geq 2$ and $d \geq 2$, there are non-linear eroders.

In the same vein we call D a *top-eroder* if toptime of every island is finite. We call D a linear top-eroder if there is a constant c such that toptime of any island x does not exceed $c(\mathrm{diam}(x)+1)$. Of course, toptime of any island cannot exceed its lifetime, so all the eroders considered by Galperin and Toom are linear top-eroders also. We hypothesize that for any non-linear top-eroder D the operator $F_r D$ is ergodic for all $r > 0$. Our Theorem 2 proves this hypothesis for one non-linear eroder, which we denote G.

In all the following text we assume that $m = d = 2$ and use letters i and j to denote the coordinates of our two-dimensional space. We associate the words "east" and "west" with the positive and negative directions of the first axis i and "north" and "south" with the positive and negative directions of the second axis j. Operator G has $n = 5$ and its neighbor vectors are

$$v_1 = (-1,0),\ v_2 = (0,1),\ v_3 = (0,0),\ v_4 = (0,-1),\ v_5 = (1,0), \tag{1.4}$$

so that the components are placed like this:

	$\mathcal{N} = x_{(0,1)}$	
$\mathcal{W} = x_{(-1,0)}$	$\mathcal{C} = x_{(0,0)}$	$\mathcal{E} = x_{(1,0)}$
	$\mathcal{S} = x_{(0,-1)}$	

Our notations are helpful because they remind $\mathcal{W}$- west, $\mathcal{N}$- north, $\mathcal{C}$- center, $\mathcal{S}$- south and $\mathcal{E}$- east. We define a function $f : \{0,1,2\}^5 \to \{0,1,2\}$ as follows:

$$f(x) \quad = \quad \begin{cases} 1 & \text{if } \mathcal{C}=2,\ \mathcal{N} \leq 1,\ \mathcal{E}=0, \\ 0 & \text{if } \mathcal{C}=1,\ \mathcal{S}=0,\ \mathcal{E}=0, \\ \mathrm{C} & \text{in all the other cases.} \end{cases} \tag{1.5}$$

In is easy to check that f is monotonic and $f(0,\ldots,0)=0$ and $f(2,\ldots,2)=2$. Notice also that

$$f(x) \quad = \quad \{\ 2 \quad \text{whenever } \mathcal{C}=\mathcal{E}=2. \tag{1.6}$$

Our G is defined as follow:

$$(Gx)_v \quad = \quad f(x_{v+v_1},\ x_{v+v_2},\ x_{v+v_3},\ x_{v+v_4},\ x_{v+v_5}),$$

where $v_1,\ldots,v_5$ are defined in (1.4) and f is defined in (1.5). Our main Theorems 1 and 2 describe behavior of G without and with one-sided random noise.

Given integer numbers $i_{\min} \leq i_{\max}$ and $j_{\min} \leq j_{\max}$, we denote by

$$[i_{\min},\ i_{\max}][j_{\min},\ j_{\max}]$$

and call a *rectangle* the following subset of $\mathbb{Z}^2$:

$$\{(i,j) : i_{\min} \le i \le i_{\max},\ j_{\min} \le j \le j_{\max}\}. \tag{1.7}$$

By $Q[i_{\min},\ i_{\max}][j_{\min},\ j_{\max}]$ we denote the island, which has twos in the set (1.7) and zeros outside it. Any island of this form is called *rectangular.*

Due to uniformity, instead of all rectangular islands, it is sufficient to deal only with those of them, for which $i_{\min} = j_{\min} = 0$ and in this case we use a simplified notation

$$Q(q,\ n) = Q[i_{\min},\ i_{\max}][j_{\min},\ j_{\max}],$$

where

$$i_{\min} = j_{\min} = 0,\ i_{\max} = q-1,\ j_{\max} = n-1.$$

Theorem 1.
a) For any rectangular island $Q(q,n)$

$$\text{lifetime}_G(Q(q,\ n)) \ = \ 2nq \quad \text{and} \quad \text{toptime}_G(Q(q,\ n)) \ = \ 2nq - n.$$

b) For any island x

$$\text{toptime}_G(x) \ \le \ \text{lifetime}_G(x) \ \le \ 2(\text{diam}(x)+1)^2.$$

The item b) of this theorem shows that G is an eroder (hence top-eroder) and the item a) shows that it is non-linear both as eroder and top-eroder.

We do not present a detailed proof of item a). Instead we show the process of evolution of a square island $Q(3,3)$ under the action of G, at the end of which it turns into "all zeros".

Figure 1 shows that the toptime of this island is 15 and lifetime is 18 in agreement with item a). It is easy to generalize this example to prove item a).

Proof of item b) of Theorem 1. Given any island x, let us denote $i_{\min}$ and $i_{\max}$ the minimal and maximal values of i for which there is j such that $x_{i,j} \ne 0$. Analogously we define $j_{\min}$ and $j_{\max}$. Then

$$x \ \prec \ Q[i_{\min},\ i_{\max}][j_{\min},\ j_{\max}].$$

Notice that

$$i_{\max} - i_{\min} \ \le \ \text{diam}(x) \quad \text{and} \quad j_{\max} - j_{\min} \ \le \ \text{diam}(x).$$

Shifting this rectangular island, we turn it into $Q(q,n)$, where

$$q \le \text{diam}(x) + 1 \quad \text{and} \quad n \le \text{diam}(x) + 1.$$

Then

$$\text{lifetime}_G(x) \ \le \ \text{lifetime}_G(Q(q,n)) \ = \ 2qn \ \le \ 2(\text{diam}(x)+1)^2.$$

Theorem 1 is proved.

Theorem 2. *The operator $F_r G$ is ergodic for all $r > 0$.*

Proof. Let us denote $\mu_T = (F_r D)^T \delta_0$ or, what is the same, the restriction of our process to the time T. It is sufficient to prove that $\lim_{T\to\infty} \mu_T = \delta_2$, which due to uniformity follows from

$$\lim_{T\to\infty} \mu_T(x_{0,0} = 2) = 1,$$

or, by definition of limit,

$$\forall\, \varepsilon > 0 \;\; \exists\, t_0 \;\; \forall\, T \geq t_0 : \; \mathcal{P}(x^T_{0,0} = 2) \geq 1 - \varepsilon. \tag{1.8}$$

Since both F_r and D are monotonic, the sequence $(F_r D)^T \delta_0$ is non-decreasing, that is $(F_r D)^T \delta_0 \prec (F_r D)^{T+1} \delta_0$ for all T. Therefore the sequence $\mathcal{P}(x^T_{0,0} = 2)$ also is non-decreasing. Therefore, (1.8) follows from

$$\forall\, \varepsilon > 0 \;\; \exists\, T \; : \; \mathcal{P}(x^T_{0,0} = 2) \geq 1 - \varepsilon. \tag{1.9}$$

This is what we shall actually prove.

Proof of (1.9) uses the auxiliary variables γ^t_v defined above and generally reminds proof of Theorem 0 and the idea of "metastability". Namely, we wait for a long time until a "critical droplet" appears due to a rare coincidence. In the present case the role of droplet is played by a large enough rectangle filled with twos like that in Figure 1. However, unlike Theorem 0, the size of our droplet depends on ε and even when such a droplet appears, we have no guarantee that it will never disappear.

Besides finite rectangles, we need their infinite analogs: For any integer $i_{\min}$, $j_{\min} \leq j_{\max}$ we denote

$$[i_{\min},\, \infty][j_{\min},\, j_{\max}] = \{(i,j) \;:\; i_{\min} \leq i,\; j_{\min} \leq j \leq j_{\max}\}$$

We denote by $Q[i_{\min},\, \infty][j_{\min},\, j_{\max}]$ the configuration, which has twos in this set and zeros outside it. Accordingly, $\delta_{[i_{\min},\, \infty][j_{\min},\, j_{\max}]}$ means the measure concentrated in this configuration. In particular, $Q[0,\, \infty][0,\, 0]$ means a configuration having twos at the positive half-axis i (including the origin) and zeros everywhere else and the measure $\delta_{[0,\, \infty][0,\, 0]}$ is concentrated in this configuration. Our first lemma gives a lower estimation of what we get after t applications of our operator to this measure.

Lemma 1. *For any natural t*

$$(F_r G)^t \; \delta_{[0,\, \infty][0,\, 0]} \succ \sum_{k=0}^{t} \binom{t}{k} \; r^k (1-r)^{t-k} \; \delta_{[-k,\, \infty][0,\, 0]}, \tag{1.10}$$

that is the measure $(F_r G)^t \; \delta_{[0,\, \infty][0,\, 0]}$ is not less than a linear combination of measures $\delta_{[-k,\infty][0,0]}$ for $k = 0, \ldots, t$ with coefficients equal to probabilities of values of k, where the random variable k has a binomial distribution with parameters r and t.

Proof. By induction. For $t = 0$ this is trivial because the left and right side coincide. In the case $t = 1$ formula (1.10) turns into

$$F_r G\ \delta_{[0,\ \infty][0,\ 0]} \succ (1-r)\ \delta_{[0,\ \infty][0,\ 0]} + r\ \delta_{[-1,\ \infty][0,\ 0]}. \tag{1.11}$$

To prove this, let us first notice that the measure $\delta_{[0,\infty][0,0]}$ is invariant for G due to (1.6). Now, to see what happens to this measure when F_r is applied, let us represent F_r using the auxiliary variables $\gamma_{i,\ j}$ as we did before. Let us consider two cases. In the case $\gamma_{-1,\ 0} = 0$, whose probability is $1-r$, the resulting measure is not less than $\delta_{[0,\ \infty][0,\ 0]}$. In the case $\gamma_{-1,\ 0} = 1$, whose probability is r, the resulting measure is not less than $\delta_{[-1,\ \infty][0,\ 0]}$. Thus (1.11) is proved. Now the general case can be proved by induction using our argument in the case $t = 1$ as the induction step. *Lemma 1 is proved.*

0	0	0	0	0
0	2	2	2	0
0	2	2	2	0
0	2	2	2	0
0	0	0	0	0

$t = 0$

0	0	0	0	0
0	2	2	1	0
0	2	2	2	0
0	2	2	2	0
0	0	0	0	0

$t = 1$

0	0	0	0	0
0	2	2	1	0
0	2	2	1	0
0	2	2	2	0
0	0	0	0	0

$t = 2$

0	0	0	0	0
0	2	2	1	0
0	2	2	1	0
0	2	2	1	0
0	0	0	0	0

$t = 3$

0	0	0	0	0
0	2	2	1	0
0	2	2	1	0
0	2	2	0	0
0	0	0	0	0

$t = 4$

0	0	0	0	0
0	2	2	1	0
0	2	2	0	0
0	2	2	0	0
0	0	0	0	0

$t = 5$

0	0	0	0	0
0	2	2	0	0
0	2	2	0	0
0	2	2	0	0
0	0	0	0	0

$t = 6$

0	0	0	0	0
0	2	1	0	0
0	2	2	0	0
0	2	2	0	0
0	0	0	0	0

$t = 7$

0	0	0	0	0
0	2	1	0	0
0	2	1	0	0
0	2	2	0	0
0	0	0	0	0

$t = 8$

0	0	0	0	0
0	2	1	0	0
0	2	1	0	0
0	2	1	0	0
0	0	0	0	0

$t = 9$

0	0	0	0	0
0	2	1	0	0
0	2	1	0	0
0	2	0	0	0
0	0	0	0	0

$t = 10$

0	0	0	0	0
0	2	1	0	0
0	2	0	0	0
0	2	0	0	0
0	0	0	0	0

$t = 11$

0	0	0	0	0
0	2	0	0	0
0	2	0	0	0
0	2	0	0	0
0	0	0	0	0

$t = 12$

0	0	0	0	0
0	1	0	0	0
0	2	0	0	0
0	2	0	0	0
0	0	0	0	0

$t = 13$

0	0	0	0	0
0	1	0	0	0
0	1	0	0	0
0	2	0	0	0
0	0	0	0	0

$t = 14$

0	0	0	0	0
0	1	0	0	0
0	1	0	0	0
0	1	0	0	0
0	0	0	0	0

$t = 15$

0	0	0	0	0
0	1	0	0	0
0	1	0	0	0
0	0	0	0	0
0	0	0	0	0

$t = 16$

0	0	0	0	0
0	1	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

$t = 17$

0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

$t = 18$

Figure 1 *Evolution of the island $Q(3,3)$ under the action of G.*

Using uniformity, we can generalize Lemma 1 as follows:

Generalization of Lemma 1. *For any integer i_0, j_0 and natural t*

$$(F_r G)^t\ \delta_{[i_0,\ \infty][j_0,\ j_0]} \succ \sum_{k=0}^{t} \binom{t}{k} r^k (1-r)^{t-k}\ \delta_{[i_0-k,\ \infty][j_0,\ j_0]}. \tag{1.12}$$

Lemma 2. *For any integer i_0 and natural n, t the result of t applications of $F_r G$ to the δ-measure concentrated in $[i_0,\ \infty][0, n-1]$ can be estimated as follows:*

$$(F_r G)^t \delta_{[i_0,\ \infty][0,n-1]} \succ \sum_{k_0,\ldots,k_{n-1}} \text{Prob}\ (k_0, \ldots, k_{n-1})\ \delta_{k_0,\ldots,k_{n-1}}, \tag{1.13}$$

where $k_0, \ldots, k_{n-1}$ are independent random variables, each having the same binomial distribution as in Lemma 1 and $\delta_{k_0,\ldots,k_{n-1}}$ is the δ-measure concentrated in the configuration x defined as follows:

$$x_{i,j} = \begin{cases} 2 & \text{if } 0 \le j \le n-1 \text{ and } i \ge i_0 - k_j, \\ 0 & \text{in all the other cases.} \end{cases} \tag{1.14}$$

Proof. Lemma 2 directly follows from the generalization of Lemma 1 and uniformity and monotonicity of our operators.

Lemma 3. *For any integer i_0 and natural $n, t \ge 1$*

$$(F_r G)^t\ \delta_{[i_0,\ \infty][0,\ n-1]} \succ (1-p)\ \delta_{[i_0-2,\ \infty][0,\ n-1]} + p\ \delta_{[i_0,\ \infty][0,\ n-1]}, \tag{1.15}$$

where $p = n(tr + 1 - r)(1-r)^{t-1}$.

Proof. It is evident that the configuration (1.14) will only decrease if some k_j decreases. Accordingly, the δ-measure concentrated in this configuration will only decrease and the total linear combination will only decrease. Thus the measure in the right side of (1.13) will only decrease if every k_j is substituted by $k_{\min} = \min(k_0, \ldots, k_{n-1})$. Then we can only further decrease our measure substituting $k_{\min}$ by 2 if $k_{\min} \ge 2$ and by 0 otherwise. Thus we obtain estimation (1.15) with any $p \ge \text{Prob}\ (k_{\min} < 2)$. It remains only to estimate Prob $(k_{\min} < 2)$. So we do:

$$\text{Prob}\ (k_{\min} < 2) =$$

$$\text{Prob}\ (\min(k_0, \ldots, k_{n-1}) < 2) = \text{Prob}\ (\exists\ j \in \{0, \ldots, n-1\} : k_j < 2) \le$$

$$n \cdot \text{Prob}\ (k_0 < 2) = n \cdot \Big(\text{Prob}\ (k_0 = 0) + \text{Prob}\ (k_0 = 1)\Big) =$$

$$n \cdot \Big((1-r)^t + tr(1-r)^{t-1}\Big) = n(tr + 1 - r)(1-r)^{t-1}.$$

Lemma 3 is proved.

We shall use Lemma 3 only with $t = 2n$. In this case

$$\text{Prob}\ (k_{\min} < 2) \ \le\ n(2nr + 1 - r)(1-r)^{2n-1}.$$

For any positive r the last expression tends to zero when $n \to \infty$. Therefore we can choose $n > 0$ such that it will be less than $1/3$. With these values of n and t, (1.15) turns into

$$(F_r G)^{2n}\ \delta_{[i_0,\ \infty][0,\ n-1]} \ \succ\ \frac{2}{3}\ \delta_{[i_0-2,\ \infty][0,\ n-1]} + \frac{1}{3}\ \delta_{[i_0,\ \infty][0,\ n-1]}. \tag{1.16}$$

Now we go back to finite rectangles.

Lemma 4. *If $q \geq 3$, then for any integer i_0 and n chosen above,*

$$(F_r G)^{2n}\ \delta_{[i_0,\ i_0+q-1][0,\ n-1]} \succ \frac{2}{3}\ \delta_{[i_0-2,\ i_0+q-2][0,\ n-1]} + \frac{1}{3}\ \delta_{[i_0,\ i_0+q-2][0,\ n-1]}. \tag{1.17}$$

Proof. What happens on the east side of our rectangular area we see from Figure 1. Even if we ignore action of F_r (which can only increase a measure), in $2n$ steps the rectangular area filled with twos shrinks only by one column. We have taken care of this by placing $i_0 + q - 2$ on the right side of (1.17), where there was i_0+q-1 on the left side. Due to our condition $q \geq 3$ and property (1.6), this loss on the right side does not undermine that random growth on the left side, which was described by our Lemmas 2 and 3. Thus, on the left side the rectangular area filled by twos advances at least by two columns with probability at least 2/3 and remains at least intact otherwise as was described by Lemma 3. *Lemma 4 is proved.*

Lemma 5. *Let us consider our process with initial measure concentrated in the island*

$$Q[i_0 - q + 1,\ i_0][0,\ n-1]$$

with that value of n which we have chosen and q such that $(1/2)^{q-2} \leq \varepsilon/2$. Then the event

$$\forall\ v = 0, 1, 2, 3, \ldots,\ \forall\ j \in [0,\ n-1]\ :x^{2nv}_{i_0-v,\ j} = 2 \tag{1.18}$$

has probability at least $1 - \varepsilon/2$.

Proof. Lemma 4 provides a lower estimation of the measure obtained from $\delta_{[i_0-q+1,\ i_0][0,\ n-1]}$ after iterative applications of $(F_r G)^{2n}$: namely, twos fill at least a rectangular area $[L,\ R][0,\ n-1]$ where R initially equals i_0 and deterministically decreases by one at every application of $(F_r G)^{2n}$ and L initially equals $i_0 - q + 1$ and then performs a random walk: at every application of $(F_r G)^{2n}$ it remains where it was with probability 1/3 and decreases by two with probability 2/3. So $R - L$ performs a random walk in which it starts with $q - 1$ and then at every step (i.e. every application of $(F_r G)^{2n}$) decreases by 1 with probability 1/3 and increases by 1 with probability 2/3. In the spirit of the historical problem of "gambler's ruin", we call it a *ruin* if $R-L$ ever becomes less than 2. In our case the probability of ruin is $(1/2)^{q-2}$. But we have chosen q such that $(1/2)^{q-2} \leq \varepsilon/2$. Now notice that absence of ruin assures the event (1.18). *Lemma 5 is proved.*

Now let us prove (1.9). Remember that n and q are already chosen. We take $T = 2n \cdot U$, where natural U is so large that

$$\left(1 - r^{nq}\right)^U \leq \frac{\varepsilon}{2}. \tag{1.19}$$

For every $u = 1, \ldots, U$ we define an event E_u as follows:

$$E_u = \{\gamma\ :\ \forall\ i \in [U - u - q + 1,\ U - u],\ \forall\ j \in [0,\ n-1]\ :\ \gamma^{2nu}_{i,\ j} = 1\}.$$

For the same values of u we denote also events

$$E'_u = E_u \cap \bigcap_{w=1}^{u-1} (\text{ not } E_w).$$

If $u = 1$, this means $E'_1 = E_1$. Generally E'_u means that E_u is the first of $E_1, \ldots, E_U$ that comes true. Also let us denote

$$E_{bad} = \bigcap_{u=1}^{U} (\text{ not } E_u),$$

which means bad luck: none of E_u happens. Notice that $P(E_u) = r^{qn}$, therefore $P(\text{ not } E_u) = 1 - r^{qn}$ and the probability that none of E_u occurs is $(1 - r^{qn})^U$ Thus, due to (1.19), $\mathcal{P}(E_{bad}) \leq \varepsilon/2$.

It is evident that the events $E'_1, \ldots, E'_U$ and E_{bad} are pairwise incompatible and together cover all Γ. Suppose that E'_u takes place. Then, applying Lemma 5, we conclude that the event

$$\forall\, v = 0, 1, 2, 3, \ldots,\ \forall\, j \in [0,\ n-1]\ :\ x^{2nu+2nv}_{U-u-v} = 2 \tag{1.20}$$

has probability at least $1 - \varepsilon/2$. Substituting $v = U - u$ and $j = 0$ into (1.20), we get the event

$$x^{2nU}_{0,0} = 2, \tag{1.21}$$

whose conditional probability also is not less than $1 - \varepsilon/2$.

Thus the probability of (1.21) is not less than

$$1 - \left(1 - \frac{\varepsilon}{2}\right)\left(1 - \frac{\varepsilon}{2}\right) \geq 1 - \varepsilon.$$

Formula (1.9) is proved, so Theorem 2 is proved also.

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