Law of Large Numbers for Non-Local Functions on Probabilistic Cellular Automata

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Abstract

We prove a multi-dimensional version of the law of large numbers for invariant measures of a large class of probabilistic cellular automata, whose transition probabilities satisfy some inequalities, which are known to assure their ergodicity. In some non-ergodic cases analogous results have been obtained for local functions. We deal with a larger class of functions, which includes some non-local ones.

Key words: law of large numbers, probabilistic cellular automata.

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A multi-dimensional law of large numbers.

Proofs of the Law of Large Numbers or LLN for interacting random processes are non-trivial even when applied to special cases [1, 2]. Some deal with applications [3, 4, 7]. Several theorems and propositions proved in Liggett's well-known books [5, 6] seem to be the most important general results in this area. Especially relevant to our cause are proposition 4.18 on page 40, proposition 2.16 on page 143 and open problem 7 on page 178 of [5] and theorem B52 on page 23 of [6]. However, all the results of [5, 6] are about processes with continuous time. It would be great to transform them into analogous discrete time results, but till now it has not been done (unless this article is counted to this end). We are not aware of any algorithm capable of transforming arguments pertaining to continuous time processes with local interaction into discrete time arguments; every instance needs human intelligence. This article's contribution is to prove some version of LLN for invariant measures of a large class of probabilistic cellular automata or p.c.a. and some non-local functions in a simple straightforward way. Some of our arguments are similar to those used in [8, 9] and we shall mention these works when appropriate.

We denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{Z}$ the set of integer numbers, by $\mathbb{Z}_+$ the set of non-negative integer numbers, and by $\mathbb{Z}'_+$ the set of positive integer numbers also known as natural numbers. By $\#(\cdot)$ we mean cardinality. We choose a non-empty finite set $\mathcal{A}$ called *alphabet* and a natural number d called *dimension*. The *space* of our process is $\mathbb{Z}^d$ and its *configuration space* is $\Omega = \mathcal{A}^{\mathbb{Z}^d}$. Every *configuration* $x \in \Omega$ is determined by its *components* $x_v \in \mathcal{A}$ for all $v \in \mathbb{Z}^d$.

For any $v \in \mathbb{Z}^d$ and any $a \in \mathcal{A}$ we call a *basic cylinder* with a *locus* $\{v\}$ the set of those configurations, which have a at the place v. We may denote this set by $\{x \in \Omega : x_v = a\}$. Any non-empty intersection of several basic cylinders is called a *thin cylinder* with a *locus*, which is the union of their loci. We may denote any thin cylinder by $\{x \in \Omega : x_L = a_L\}$, where L is any non-empty finite subset of $\mathbb{Z}^d$, called the *locus* of this thin cylinder, x_L is the restriction of x to L and a_L is an arbitrary element of $\mathcal{A}^L$. Elements of the algebra generated by thin cylinders are called *cylinders*. A finite set $L \subset \mathbb{Z}^d$ is called a *locus* of a cylinder C if it is the minimal set with the following property: for any configuration $x \in \Omega$ it is sufficient to know its restriction to L to decide, whether x belongs to C. We denote by $\mathcal{M}$ the set of normalized measures on the σ-algebra generated by cylinders. By convergence in $\mathcal{M}$ we mean convergence on all cylinders.

We may add and subtract elements of $\mathbb{Z}^d$ as vectors. It this spirit, for any set $S \subset \mathbb{Z}^d$ and any vector $v \in \mathbb{Z}^d$ we denote $S + v = \{w + v : w \in S\}$ and $S - v = \{w - v : w \in S\}$. Based on this, given any vector $v \in \mathbb{Z}^d$:

• for any configuration $x \in \Omega$ we denote by $x \oplus v$ the translation of x at v, that is another configuration defined by the rule $\forall\, w \in \mathbb{Z}^d : (x \oplus v)_w = x_{w-v}$;

• for any set $C \subset \Omega$ we denote by $C \oplus v$ the translation of C at v, that is another subset of Ω defined by the rule $C \oplus v = \{x \oplus v : x \in C\}$;

• for any real function $f : \Omega \to \mathbb{R}$ we denote by $f \oplus v$ another function from Ω to $\mathbb{R}$ defined by the rule $(f \oplus v)(x) \equiv f(x \oplus v)$;

• for any $\mu \in \mathcal{M}$ we denote by $\mu \oplus v$ another measure such that $(\mu \oplus v)(C) = \mu(C \oplus v)$ for any cylinder C; we call a measure $\mu \in \mathcal{M}$ *uniform* if $\mu \oplus v = \mu$ for all v.

As soon as we want to prove a law of large numbers for some measures on an infinite-dimensional product $\mathcal{A}^{\mathbb{Z}^d}$, we need first to define it because the classical law of large numbers was stated for sequences of random variables. One way to manage the multi-dimensionality is to take average over $0 \le y \le x$, that is $0 \le y_i \le x_i$ for all $i = 1, \ldots, d$, and then make all x_i tend to ∞, as was done in theorem 4.9 and some other statements in chapter 1 of [5]. The definition, which we use, is stronger and similar to that of [9].

Main definition. Suppose that we have a uniform measure $\mu \in \mathcal{M}$ and a function $f : \Omega \to \mathbb{R}$ such that f has a mathematical expectation according to μ. We say that a *d-dimensional law of large numbers* is true for μ and f if there is a number H such that for any natural k, any k different vectors $v_1, \ldots, v_k \in \mathbb{Z}^d$ and any $\varepsilon > 0$

$$\mu\left(\left| \frac{1}{k} \sum_{i=1}^{k} \Big((f \oplus v_i) - E(f \oplus v_i) \Big) \right| \ge \varepsilon \right) \le \frac{H}{k \cdot \varepsilon^2}.$$

Another strength of this definition is its right side, which is more explicit than usual. Throughout this article var_m and covar_m mean variance and covariance according to any measure m.

Main lemma. Suppose that we have a uniform measure $\mu \in \mathcal{M}$ and a function

$f : \Omega \to \mathbb{R}$ such that all $\mathrm{covar}_\mu(f,\ f \oplus v)$ exist and

$$\sum_{v \in \mathbb{Z}^d} \Big| \mathrm{covar}_\mu(f,\ f \oplus v) \Big| < \infty. \tag{1}$$

Then the d-dimensional law of large numbers is true for μ and f.

Proof. Let us estimate:

$$\mathrm{var}_\mu \left(\sum_{i=1}^{k} f \oplus v_i \right) =$$

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \mathrm{covar}_\mu(f \oplus v_i,\ f \oplus v_j) \le$$

$$\sum_{i=1}^{k} \sum_{j=1}^{k} |\mathrm{covar}_\mu(f \oplus v_i,\ f \oplus v_j)| \tag{2}$$

Since all v_j are different, we may estimate (2) as follows:

$$\sum_{i=1}^{k} \sum_{j=1}^{\infty} |\mathrm{covar}_\mu(f \oplus v_i,\ f \oplus v_j)| \le k \cdot \sum_{v \in \mathbb{Z}^d} |\mathrm{covar}_\mu(f,\ f \oplus v)|.$$

According to our assumption, this sum converges and may be denoted by H. Thus

$$\mathrm{var}_\mu \left(\sum_{i=1}^{k} (f \oplus v_i) \right) \le k \cdot H$$

for all natural k and all different vectors $v_1, \ldots, v_k$. *Now our main lemma follows from Tchebyshev inequality.*

Probabilistic cellular automata and statement of the theorem.

Any map $P : \mathcal{M} \to \mathcal{M}$ is called an *operator*. A measure $\mu \in \mathcal{M}$ is called *invariant* for P if $P\mu = \mu$. Existence of at least one invariant measure has been proved for a very large class of operators (see e.g. [11]), including all those considered here. As usual, we call P *ergodic* if it has a unique invariant measure μ such that $P^n \nu$ tends to μ for any initial measure ν.

Probabilistic cellular automata or p.c.a. for short, as defined here, are linear operators $P : \mathcal{M} \to \mathcal{M}$, parametrizied by a non-empty finite set $N \subset \mathbb{Z}^d$ called *neighborhood* and non-negative numbers $\theta(b\,|\,a)$, where $b \in \mathcal{A}$ and $a \in \mathcal{A}^N$, subject to condition

$$\forall\ a \in \mathcal{A}^N : \sum_{b \in \mathcal{A}} \theta(b\,|\,a) = 1.$$

Given these parameters, the following formula (3) defines a generic p.c.a. as it expresses the value of $P\mu$ at an arbitrary thin cylinder as a linear combination of values of μ at several thin cylinders:

$$(P\mu)(y_S = b_S) \ = \sum_{a_{S+N} \in \mathcal{A}^{S+N}} \mu(x_{S+N} = a_{S+N}) \prod_{i \in S} \theta(b_i\,|\,a_{i+N}) \tag{3}$$

for any finite set $S \subset \mathbb{Z}^d$ and any $b_i \in \mathcal{A}$ for all $i \in S$.

For every $b \in \mathcal{A}$ we denote by $sure(b)$ the minimum of $\theta(b\,|\,a)$ over all $a \in \mathcal{A}^N$. Also we denote

$$unsure = 1 - \sum_{b \in \mathcal{A}} sure(b).$$

Given a p.c.a. P, we call its *memory* and denote by $mem(P)$ this:

$$mem(P) = \#(N) \cdot unsure.$$

It is known that any p.c.a. P with $mem(P) < 1$ is ergodic. The idea of a proof was explained e.g. in [10, section 5.2]. See also the seminal work [12].

We call a function $f : \Omega \to \mathbb{R}$ *local* if there is a finite set $S \subset \mathbb{Z}^d$ such that the value of $f(x)$ is in fact determined by those components of x, which are placed in S; in other words, $f(x) \equiv f(x_S)$. The minimal S with this property (which exists and is unique for any local function) is called the *locus* of this function and denoted by $locus(f)$. We use the Euclidean norm $\|\cdot\|$ on our discrete space $\mathbb{Z}^d$ and containing it continuoua space $\mathbb{R}^d$ and denote for any finite set $S \subset \mathbb{Z}^d$

$$\rho(S) = \max_{v \in S} \|v\|.$$

Theorem. Suppose that we have a p.c.a. P with $mem(P) < 1$. Suppose also that we have a function $f : \Omega \to \mathbb{R}$, which can be represented as a series

$$f = \sum_{i=1}^{\infty} f_i, \tag{4}$$

where

$$\sum_{i=1}^{\infty} i^d \cdot \sup |f_i| < \infty, \tag{5}$$

every f_i is local and there is a number L such that

$$\forall\ i : \rho(locus(f_i)) \leq L \cdot i. \tag{6}$$

Then P has a unique invariant measure, which we denote by inv , function f has a mathematical expectation according to inv , and the d-dimensional law of large numbers is true for the measure inv and function f .

Proof.

As we said before, under our conditions P is ergodic and therefore has a unique invariant measure inv . Since inv is unique, it is uniform. From (5) f is bounded and therefore has a mathematical expectation according to any measure on Ω . The bulk of this article is the proof of the d-dimensional law of large numbers. We assume that $\#(N) > 1$ and $unsure > 0$, because thus excluded special cases are easy. Also, without loss of generality we may assume that

$$\forall\ i : \rho(locus(f_i)) \geq 1. \tag{7}$$

Auxiliary statements.

Starting here we fix a certain p.c.a. P and a function f , which satisfy the conditions of the theorem. A *constant* means a number, which depend only on P and f . Since $unsure$ is positive, $mem(P)$ is also positive. Using this, for any real number X we denote

$$\text{expmem}(X) = (mem(P))^X.$$

Proposition 1. For any p.c.a. P with $mem(P) < 1$, any cylinders $C,\ D \subset \Omega$ and any $v \in \mathbb{Z}^d$

(a) $|\text{covar}_{inv}(I_C,\ I_D \oplus v)| \le 1/4$;

$$\text{(b) if } \|v\| \ge 2\rho(locus(C)) + 2\rho(locus(D)) + 4\rho(N), \tag{8}$$

then $|\text{covar}_{inv}(I_C,\ I_D \oplus v)| \le$

$$\Big(\#(locus(C)) + \#(locus(D))\Big) \cdot \text{expmem}\left(\frac{\|v\|}{4\rho(N)}\right), \tag{9}$$

where I_C and I_D are the indicator functions of C and D.

Proof of proposition 1. The item (a) follows from the fact that

$$\text{covar}_{inv}(I_C,\ I_D \oplus v) = inv(C \cap D \oplus v) - inv(C) \cdot inv(D \oplus v).$$

The maximum of this expression is $1/4$ and takes place when $C = D \oplus v$ and $inv(C) = 1/2$.

Let us prove the item (b). The following will be called the *big space*:

$$\Big(\mathcal{A} \times \mathcal{A} \times \{\text{true},\ \text{false}\}\Big)^{\mathbb{Z}^d \cdot \mathbb{Z}_+}, \tag{10}$$

Here the first and second factors are denoted by x and x' and the third factor is denoted by old. The variable $old(v,\ t)$ is Boolean and takes values "true" and "false", to which we apply the usual operations $\vee$ and $\wedge$. Thus for every point $(v, t) \in \mathbb{Z}^d \cdot \mathbb{Z}_+$ we have

$$x(v,\ t),\ x'(v,\ t) \in \mathcal{A} \text{ and } old(v,\ t) \in \{\text{true},\ \text{false}\}.$$

We call a point (v, t) *old-fashioned* if $old(v, t) = \text{true}$ and *new-fashioned* otherwise. Informally, an old-fashioned point still keeps some memory of the initial condition, while state of a new-fashioned point is completely determined by the random noise of subsequent time steps. Since $mem(P) < 1$, the old-fashioned points die out as time t tends to ∞.

We shall construct a probability distribution on the big space, which we call the *big process*. The big process will be induced by a certain *auxiliary distribution* $\mathcal{P}$ of *auxiliary variables* with a certain *auxiliary map*. Let us describe the auxiliary variables and their distribution. They include:

The initial condition denoted by x_{init} ; it is distributed according to inv independently from all the other auxiliary variables.

For every point (v, t) , where $v \in \mathbb{Z}^d$ and $t \in \mathbb{Z}'_+$, we introduce the following auxiliary variables, all of which are independent from each other and from x_{init} . We denote by $\mathcal{M}_{\mathcal{A}}$ the set of probability distributions on $\mathcal{A}$.

- Identically distributed random variables $\eta_a(v,\ t) \in \mathcal{M}_{\mathcal{A}}$ indexed by $a \in \mathcal{A}^N$ and distributed as follows:

$$\mathcal{P}\left(\eta_a(v,\ t) = c\right) = \frac{\theta(c|a) - sure(c)}{unsure} \quad \text{for any } \ c \in \mathcal{A}.$$

The denominator of this fraction is not zero because we have excluded the case $unsure = 0$.

- A variable $assign(v,\ t)$, whose set of values is $\mathcal{A} \cup \{free\}$, where $free$ is an especially coined state not belonging to $\mathcal{A}$. This variable is distributed as follows: For any $a \in \mathcal{A}$ the probability that $assign(v,\ t) = a$ is $sure(a)$; finally, $assign(v,\ t) = free$ with the remaining probability $unsure$.

Now let us describe the auxiliary map. It is defined by induction in t .

Base of induction:

$$\left.\begin{aligned} x(v,\ 0) &= x_{\text{init}}(v), \\ old(v,\ 0) &= \text{true} \end{aligned}\right\} \text{ for all } v \in \mathbb{Z}^d.$$

The t-th *induction step*. Suppose that $x(v,\ t-1)$ and $old(v,\ t-1)$ are already determined as functions of the auxiliary variables for all $v \in \mathbb{Z}^d$. Then we determine $x(v,\ t)$ and $old(v,\ t)$ for all v , proceeding in two stages.

Stage 1.

$$x'(v,\ t) = \eta_a(v,\ t), \text{ where } a = x(v+N,\ t-1).$$

Stage 2.

$$\left.\begin{array}{l} \bullet \text{ If } assign(v,\ t) \in \mathcal{A}\text{, then we define} \\ \qquad x(v,\ t) = assign(v,\ t) \text{ and } old(v,\ t) = \text{false.} \\ \bullet \text{ If } assign(v,\ t) = free\text{, then we define} \\ \qquad\qquad x(v,\ t) = x'(v,\ t) \text{ and} \\ \qquad\qquad old(v,\ t) = \bigvee_{w \in N} old(t-1,\ v+w). \end{array}\right\}$$

Thus the auxiliary map is defined. Thereby the big distribution on the big space (10) is defined also: it is induced by the distribution of the auxiliar variables with the auxiliary map. We shall use the same letter $\mathcal{P}$ for values of the big distribution.

For every t let us denote by $\mathcal{P}^t$ the restriction of $\mathcal{P}$ to the time level t. Notice that for any t the restriction of $\mathcal{P}^t$ to the marginal x coincides with inv.

Let us observe also that the marginal *old* of the big process does not depend on the other marginals and in fact is an oriented site percolation model. Its vertices are $(v,\ t)$, where $v \in \mathbb{Z}^d$ and $t \in \mathbb{Z}_+$. Its edges go from any vertex $(v,\ t)$ to the vertices $(v-i,\ t+1)$ for all $i \in N$. They are always open in this direction and always closed in the opposite direction. Every vertex with $t > 0$ is open with a probability $unsure$ and closed with the remaining probability $1 - unsure$ independently from other vertices. A path from $(v',\ t')$ to $(v,\ t)$ is a sequence of alternating vertices and edges

$$(v_0,\ t_0) \to (v_1,\ t_1) \to \cdots \to (v_n,\ t_n), \tag{11}$$

where arrows mean edges, passed in those directions in which they are always open, $(v_0,\ t_0) = (v',\ t')$, $(v_n,\ t_n) = (v,\ t)$ and every edge goes from the vertex, which preceeds it to the vertex, which succeeds it in the sequence (11). We call a path *open* if all its vertices are open.

Lemma 1.

(a) $\mathcal{P}\Big((v,t)$ is old-fashioned $\Big) \leq \text{expmem}(t)$.

(b) For any finite set $S \subset \mathbb{Z}^d$

$$\mathcal{P}\Big(\exists\, v \in S : (v,\ t) \text{ is old-fashioned } \Big) \leq \#(S) \cdot \text{expmem}(t).$$

Proof. First we prove item (a). Any point $(v,\ t)$ is old-fashioned if and only if there is an open path from the level zero to $(v,\ t)$ in our percolation model. There are $(\#(N))^t$ paths from the level zero to (v,t) and each of them is open with a probability $unsure^t$. So the probability $\mathcal{P}$ that at least one of them is open does not exceed

$$\Big(\#(N)\Big)^t \cdot unsure^t = \text{expmem}(t).$$

Item (a) of lemma 1 is proved. Item (b) follows immediately.

Lemma 2. (Similar to lemma 1 in [8].) Suppose that we have a measure m on some space and events

$$A = A_1 \cup A_2 \text{ and } B = B_1 \cup B_2,$$

where A_1 and B_1 are independent according to m . Then

$$|\text{covar}_m(I_A,\ I_B)| \leq m(A_2) + m(B_2).$$

Proof. First notice that for any events $A,\ B$

$$\text{covar}_m(I_A,\ I_B)\ =\ m(A \cap B) - m(A) \cdot m(B). \tag{12}$$

Now, on one hand

$$\begin{aligned}
& m(A_1 \cap B_1) \leq m(A \cap B) = \\
& m\Big((A_1 \cup A_2) \cap (B_1 \cup B_2)\Big) = \\
& m\Big((A_1 \cap B_1) \cup (A_1 \cap B_2) \cup (A_2 \cap B)\Big) \leq \\
& m(A_1 \cap B_1) + m(A_2) + m(B_2).
\end{aligned} \tag{13}$$

On the other hand

$$m(A_1)\cdot m(B_1)\le m(A)\cdot m(B)=$$
$$m(A_1\cup A_2)\cdot m(B_1\cup B_2)=$$
$$\Big(m(A_1)+m(A_2\setminus A_1)\Big)\cdot\Big(m(B_1)+m(B_2\setminus B_1)\Big)=$$
$$m(A_1)\cdot m(B_1)+m(A_1)\cdot m(B_2\setminus B_1)+m(A_2\setminus A_1)\cdot m(B)\le$$
$$m(A_1)\cdot m(B_1)+m(A_2)+m(B_2). \tag{14}$$

Lemma 2 follows from (12) and estimations (13) and (14) .

Now for any cylinders $C,\ D\subset\Omega$ we define the events:

$$\left.\begin{array}{lll} A &= (C,t); & \\ A_1 &= (C,t) & \text{and all the points } (w,\ t)\text{, where} \\ & & w\in \mathit{locus}(C)\text{, are new-fashioned;} \\ A_2 &= A\setminus A_1; & \\ B &= (D\oplus v,t); & \\ B_1 &= (D\oplus v,t) & \text{and all the points } (w,\ t)\text{, where} \\ & & w\in \mathit{locus}(D)+v\text{, are new-fashioned;} \\ B_2 &= B\setminus B_1; & \end{array}\right\} \tag{15}$$

where t is a natural parameter, whose value will be chosen later.

Lemma 3. Let $A,\ A_1,\ A_2,\ B,\ B_1,\ B_2$ be defined by (15) . Then for any natural t

$$\mathcal{P}(A_2)\le \#(\mathit{locus}(C))\cdot \mathrm{expmem}(t),\quad \mathcal{P}(B_2)\le \#(\mathit{locus}(D))\cdot \mathrm{expmem}(t).$$

Proof follows from lemma 1.

Now let us choose the value of t as follows:

$$t=]t'[-1,\quad \text{where}\quad t'=\frac{\|v\|-\rho(\mathit{locus}(C))-\rho(\mathit{locus}(D))}{2\rho(N)}, \tag{16}$$

where the denominator is not zero because we have assumed that $\#(N)\ge 2$. In other words, t is the greatest integer number, which is less than t' . Notice that due to (8)

$$1 \le t' - 1 \le t < t'. \tag{17}$$

Lemma 4. Let $A_1 = C$ and $B_1 = D \oplus v$ be defined by (15) and t defined by (16) . Then A_1 and B_1 are independent according to $\mathcal{P}$.

Proof. For every set $(S,\ t)$, where $S \subset \mathbb{Z}^d$, we define a set $\pi(S,\ t)$ as follows:

$$\pi(S,\ t) = (S + N,\ t - 1).$$

For any set $(S,\ t)$ and any $k = 0,\ 1,\ 2, \ldots, t$ we define $\pi^k(S,\ t)$ by induction: first $\pi^0(S,\ t) = (S,\ t)$, then $\pi^{k+1}(S,\ t) = \pi(\pi^k(S,\ t))$. Finally we denote

$$\Pi(S,\ t)\ =\ \bigcup_{k=0}^{t}\ \pi^k(S,\ t).$$

Evidently, whether the event A_1 takes place or not, depends only on those auxiliary variables, which are placed in $\Pi(locus(C),\ t)$; analogously, whether the event B_1 takes place or not, depends only on those auxiliary variables, which are placed in $\Pi(locus(D) + v,\ t)$. To prove that these sets do not intersect, let us denote by S_1 the set of space components of elements of $\Pi(locus(C),\ t)$ and by S_2 the set of space components of elements of $\Pi(locus(D) + v,\ t)$. Then

$$\rho(S_1) \le \rho(C) + t \cdot \rho(N) \text{ and } \rho(S_2 - v) \le \rho(D) + t \cdot \rho(N).$$

If S_1 and S_2 have a common element, then by the triangle inequality

$$\|v\|\ \le\ \rho(locus(C)) + \rho(locus(D)) + 2t\rho(N),$$

whence $t \ge t'$, which contradicts (17) . *Lemma 4 is proved.*

Now let us prove the item (b) of proposition 1. From lemma 4, the events A_1 and B_1 defined in (15) are independent according to $\mathcal{P}$, provided t is defined by (16) . This allows us to use lemma 2 to conclude that

$$|\mathrm{covar}_{\mathcal{P}}(I_A,\ I_B)| \le \mathcal{P}(A_2) + \mathcal{P}(B_2),$$

where A, B, A_2 and B_2 are also defined by (15) . Then, using lemma 3, we get

$$|\mathrm{covar}_{\mathcal{P}}(I_A,\ I_B)| \le \Big(\#(locus(C)) + \#(locus(D))\Big) \cdot \mathrm{expmem}(t).$$

Since $mem(P) < 1$ and $t \ge t' - 1$, this implies

$$|\mathrm{covar}_{\mathcal{P}}(I_A,\ I_B)| \le \Big(\#(locus(C)) + \#(locus(D))\Big) \cdot \mathrm{expmem}(t' - 1).$$

Substituting here the expression (16) for t' , we get

$$|\mathrm{covar}_{\mathcal{P}}(I_A,\ I_B)| \ \le\ \Big(\#(locus(C)) + \#(locus(D))\Big) \cdot$$

$$\mathrm{expmem}\left(\frac{\|v\| - \rho(locus(C)) - \rho(locus(D))}{2\rho(N)} - 1\right). \tag{18}$$

But from (8)

$$\|v\| - \rho(locus(C)) - \rho(locus(D)) - 2\rho(N)\ \ge\ \frac{\|v\|}{2}.$$

Hence, from (18)

$$|\mathrm{covar}_{\mathcal{P}}(I_A,\ I_B)| \le \Big(\#(locus(C)) + \#(locus(D))\Big) \cdot \mathrm{expmem}\left(\frac{\|v\|}{4\rho(N)}\right).$$

Hence follows the item (b) of proposition 1.

Example. The following example should clarify our way to prove proposition 1. Let us assume that our alphabet has only two elements 0 and 1. Let us denote $n = \#(N)$ and let us have n vectors $v_1, \ldots, v_n \in \mathbb{Z}^d$ and a non-constant function $\phi : \mathcal{A}^n \to \mathcal{A}$. This is sufficient to define a *deterministic cellular automaton*, that is a map $D : \Omega \to \Omega$ defined as follows:

$$\forall\ x \in \Omega,\ v \in \mathbb{Z}^d : (Dx)_v = \phi(x_{v+v_1}, \ldots, x_{v+v_n}).$$

Now let us transform D into a p.c.a. by adding some random noise parametrized by two non-negative numbers $sure(0)$ and $sure(1)$, whose sum does not exceed 1. The time step of our p.c.a. P consists of two parts, deterministic and random: first, given configuration x at time t , for every $v \in \mathbb{Z}^d$ the v -th component at time

$t+1$ is assigned the value $D(x_{v+v_1}, \ldots, x_{v+v_n})$ (as in the deterministic case). Then with probability $sure(0)$ it is assigned the value 0, with probability $sure(1)$ it is assigned the value 1 and with the remaining probability $unsure = 1 - sure(0) - sure(1)$ it keeps the value $D(x_{v+v_1}, \ldots, x_{v+v_n})$. In this case $mem(P) = n \cdot unsure$ and our main condition $mem(P) < 1$ is equivalent to

$$sure(0) + sure(1) > 1 - \frac{1}{n}.$$

Now to continue with our argument. Let us call a real function on Ω a *step function* if it takes at most two different values. For any real function we call its *span*

$$\operatorname{span}(f) = \sup f - \inf f.$$

Proposition 2. For any p.c.a. P with $mem(P) < 1$, any local step functions g, h and any $v \in \mathbb{Z}^d$

(a) $|\operatorname{covar}(g,\ h \oplus v)_{inv}| \ \leq\ \frac{1}{4} \cdot \operatorname{span}(g) \cdot \operatorname{span}(h);$

(b) if $\|v\| \ \geq\ 2\rho(locus(g)) + 2\rho(locus(h)) + 4\rho(N),$ then

$$|\operatorname{covar}_{inv}(g,\ h \oplus v)| \ \leq \operatorname{span}(g) \cdot \operatorname{span}(h) \times$$

$$\Big(\#(locus(g)) + \#(locus(h))\Big) \cdot \operatorname{expmem}\left(\frac{\|v\|}{4\rho(N)}\right). \tag{19}$$

Proof of proposition 2. Both statements (a) and (b) follow from proposition 1, linearity of covariance in both arguments and the fact that any local step function equals a constant plus an indicator function of some cylinder multiplied by another constant. *Proposition 2 is proved.*

Lemma 5. Any local function $\phi : \Omega \to \mathbb{R}$ can be represented as a sum of several step functions, whose loci belong to the locus of ϕ, and the sum of whose spans equals the span of ϕ.

Proof. Since ϕ is local, it takes only a finite set of different values. Let $c_0 < c_1 < \cdots < c_n$ be the complete list of values of ϕ. We may assume that ϕ is defined on a finite set $D = \mathcal{A}^L$ and represent this set as $D = D_0 \cup D_1 \cup \cdots \cup D_n$, where every $D_k = \{x \in D : \phi(x)\} = c_k$. Now we can represent ϕ as

$$\phi = \phi_0 + \cdots + \phi_n,$$

where

$$\phi_0 \equiv c_0 \ \text{ and } \ \phi_k(x) = \begin{cases} c_k - c_{k-1} & \text{if } \ x \in D_k \cup \cdots \cup D_n \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in D$ and $k = 1, \ldots, n$. Evidently, all ϕ_k are step functions and the sum of their spans equals the span of ϕ . *Lemma 5 is proved.*

Proposition 3. For any local functions $g,\ h$ and for any $v \in \mathbb{Z}^d$

$$\text{(a) } |\text{covar}_{inv}(g,\ h \oplus v)| \ \leq \ \frac{1}{4} \cdot \text{span}(g) \cdot span(h);$$

$$\text{(b) if } \|v\| \ \geq \ 2\rho(locus(g)) + 2\rho(locus(h)) + 4\rho(N),$$

$$\text{then } |\text{covar}_{inv}(g,\ h \oplus v)| \ \leq \ \text{span}(f) \cdot \text{span}(g) \times$$

$$\Big(\#(locus(g)) + \#(locus(h))\Big) \cdot \text{expmem}\left(\frac{\|v\|}{4\rho(N)}\right). \tag{20}$$

Proof of proposition 3. The item (a) follows from the item (a) of proposition 2. To prove (b), we apply lemma 5 to represent

$$g = g_0 + \cdots + g_m \text{ and } h = h_0 + \cdots + h_n,$$

where all g_i and h_j are step functions and

$$\text{span}(g) = \sum_{i=0}^{m} \text{span}(g_i) \qquad \text{span}(h) = \sum_{j=0}^{n} \text{span}(h_j).$$

Then, using linearity of covariance, we represent

$$\left|\text{covar}_{inv}(g,\ h \oplus v)\right| =$$

$$\left|\text{covar}_{inv}\left(\sum_{i=0}^{m} g_i,\ \sum_{j=0}^{n} h_j \oplus v\right)\right| \leq$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\left|covar_{inv}\Big(g_i,\ h_j \oplus v\Big)\right|. \tag{21}$$

Since all g_i and h_j are local step functions, we can apply proposition 2 to prove that (21) does not exceed

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\text{span}(g_i)\cdot\text{span}(h_j)\times$$

$$\Big(\#(locus(g)) + \#(locus(h))\Big)\cdot\text{expmem}\left(\frac{\|v\|}{4\rho(N)}\right) =$$

$$\left(\sum_{i=0}^{m}\sum_{j=0}^{n}\text{span}(g_i)\cdot\text{span}(h_j)\right)\times$$

$$\Big(\#(locus(g)) + \#(locus(h))\Big)\cdot\text{expmem}\left(\frac{\|v\|}{4\rho(N)}\right) =$$

$$\Big(\text{span}(g)\cdot\text{span}(h)\Big)\times$$

$$\Big(\#(locus(g)) + \#(locus(h))\Big)\cdot\text{expmem}\left(\frac{\|v\|}{4\rho(N)}\right).$$

Proposition 3 is proved.

Proposition 4. For any local functions $g,\ h : \Omega \to \mathbb{R}$

$$\sum_{v\in\mathbb{Z}^d}\left|\text{covar}_{inv}(g,\ h \oplus v)\right| \leq const\cdot\text{span}(g)\cdot\text{span}(h)\cdot$$

$$\left(\Big(\rho(locus(g)) + \rho(locus(h)) + 1\Big)^d + \#(locus(g)) + \#(locus(h))\right).$$

Proof of proposition 4. Let us denote

$$r_0 = 2\rho(locus(g)) + 2\rho(locus(h)) \quad \text{and} \quad r_1 = r_0 + 4\rho(N),$$

define

$$small = \left\{ v \in \mathbb{Z}^d : \|v\| \le r_1 \right\}, \quad large = \mathbb{Z}^d \setminus small,$$

represent

$$\sum_{v \in \mathbb{Z}^d} \left| \text{covar}_{inv}(g,\ h \oplus v) \right| =$$

$$\sum_{v \in small} \left| \text{covar}_{inv}(g,\ h \oplus v) \right| + \sum_{v \in large} \left| \text{covar}_{inv}(g,\ h \oplus v) \right|$$

and estimate each sum in turn. First we estimate the sum over $v \in small$. Due to the definition of $small$, $\#(small) \le const \cdot r_0^d$, whence from item (a) of proposition 3

$$\sum_{v \in small} \left| \text{covar}_{inv}(g,\ h \oplus v) \right| \le$$

$$const \cdot \text{span}(f) \cdot \text{span}(g) \cdot \Big(\rho(locus(g)) + \rho(locus(h)) + 1 \Big)^d . \qquad (22)$$

Now we estimate the sum over $v \in large$. Due to item (b) of proposition 3, this sum does not exceed

$$\text{span}(g) \cdot \text{span}(h) \cdot \Big(\#(locus(g)) + \#(locus(h)) \Big) \cdot$$

$$\sum_{v \in large} \text{expmem} \left(\frac{\|v\|}{4\rho(N)} \right) . \qquad (23)$$

The last sum (23) does not exceed a constant times the d-dimensional integral

$$\int_{\|v\| \ge r_0} \text{expmem} \left(\frac{\|v\|}{4\rho(N)} \right) \, dv.$$

This integral can be transformed into a constant times one-dimensional integral

$$\int_{r_0}^{\infty} r^{d-1} e^{-c \cdot x} \, dx, \qquad (24)$$

where

$$c \;=\; -\frac{\ln(mem(P))}{4\rho(N)} \;>\; 0.$$

It is easy to prove that the integral (24) does not exceed $const \cdot r_0^{d-1}$. Therefore

$$\sum_{v \in large} \text{expmem}\left(\frac{\|v\|}{4\rho(N)}\right) \le const \cdot \Big(\rho(locus(g)) + \rho(locus(h)) + 1\Big)^{d-1}.$$

Finally, this implies that

$$\sum_{v \in large} \Big|\text{covar}(g,\; h \oplus v)_{inv}\Big| \le const \cdot \text{span}(g) \cdot \text{span}(h) \cdot$$

$$\Big(\rho(locus(g)) + \rho(locus(h)) + 1\Big)^{d-1} \cdot$$

$$\Big(\#(locus(g)) + \#(locus(h))\Big). \qquad (25)$$

Summing (22) and (25) , we obtain proposition 4.

Proof of the theorem. Due to the representation (4) ,

$$\sum_{v \in \mathbb{Z}^d} \Big|\text{covar}_{inv}(f,\; f \oplus v)\Big| =$$

$$\sum_{v \in \mathbb{Z}^d} \Big|\text{covar}_{inv}(\sum_{i=1}^{\infty} f_i,\; \sum_{j=1}^{\infty} f_j \oplus v)\Big| =$$

$$\sum_{v \in \mathbb{Z}^d} \Big|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\text{covar}_{inv}(f_i,\; f_j \oplus v)\Big| \le$$

$$\sum_{v \in \mathbb{Z}^d}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \Big|\text{covar}_{inv}(f_i,\; f_j \oplus v)\Big| =$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{v \in \mathbb{Z}^d} \Big|\text{covar}_{inv}(f_i,\; f_j \oplus v)\Big|.$$

Taking $g = f_i$ and $h = f_j$ in proposition 4 and using (7) , we get

$$\sum_{v \in \mathbb{Z}^d} |\mathrm{covar}_{inv}(f_i,\ f_j \oplus v)| \le const \cdot \mathrm{span}(f_i) \cdot \mathrm{span}(f_j) \cdot$$

$$\left(\Big(\rho(locus(f_i)) + \rho(locus(f_j))\Big)^d + \#(locus(f_i)) + \#(locus(f_j)) \right). \quad (26)$$

Due to (6) , the bracket (26) does not exceed $const \cdot (i^d + j^d)$. Therefore

$$\sum_{v \in \mathbb{Z}^d} \Big|\mathrm{covar}_{inv}(f,\ f \oplus v)\Big| \le const \cdot \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \cdot \mathrm{span}(f_i) \cdot \mathrm{span}(f_j) \cdot \Big(i^d + j^d\Big).$$

Due to (5) , this double series converges. Denoting its sum by H , we get

$$\mathrm{var}_{inv} \left(\sum_{i=1}^{k} (f \oplus v_i) \right) \le k \cdot H.$$

Hence our theorem follows from the main lemma.

To apply our theorem to a given function f we have to expand f into a series (4) subject to conditions (5) and (6) , but it may be unclear how to do it even when it is possible. The following sufficient condition may be helpful.

For any $v \in \mathbb{Z}^d$ let us denote by $(x_v,\ z_{not(v)})$ the configuration, whose value at v is x_v and values at all the other elements of $\mathbb{Z}^d$ form a configuration $z_{not(v)}$ on $\mathbb{Z}^d \setminus \{v\}$. Given a function $f : \Omega \to \mathbb{R}$, for every $v \in \mathbb{Z}^d$, we denote

$$impact(f\,|\,v) = \sup |f(x_v,\ z_{not(v)}) - f(y_v,\ z_{not(v)})|,$$

where the supremum is taken over all $x_v,\ y_v \in \mathcal{A}$ and all $z_{not(v)} \in \mathcal{A}^{\mathbb{Z}^d \setminus \{v\}}$.

Proposition 5. Given a function $f : \Omega \to \mathbb{R}$ for which there is a number K such that

$$\forall\ v \in \mathbb{Z}^d : impact(f\,|\,v) \le K \cdot \|v\|^{-(2d+1)}. \quad (27)$$

Then f can be represented as a series (4) subject to conditions (5) and (6) .

Proof of proposition 5. For every $S \subset \mathbb{Z}^d$, we denote

$$impact(f\,|\,S) = \sup |f(x_S,\ z_{not(S)}) - f(y_S,\ z_{not(S)})|,$$

where the supremum is taken over all $x_S,\ y_S \in \mathcal{A}^S$ and all $z_{not(S)} \in \mathcal{A}^{\mathbb{Z}^d \setminus S}$. Notice that

$$impact(f\,|\,S) \le \sum_{v \in S} impact(f\,|\,v).$$

Let us choose an arbitrary element of $\mathcal{A}$ and denote it by 0. Given a configuration $x \in \Omega$ and a set $S \subset \mathbb{Z}^d$, we denote by $x\,|_S$ the configuration defined as follows:

$$\forall\, v \in \mathbb{Z}^d : x\,|_S\,(v) = \begin{cases} x(v) & \text{if } v \in S, \\ 0 & \text{if } v \notin S. \end{cases}$$

For any $f : \Omega \to \mathbb{R}$ and any $S \subset \mathbb{Z}^d$ we define another function $f\,|_S$ as follows:

$$\forall\, x \in \Omega : f\,|_S\,(x) = f(x\,|_S).$$

Evidently

$$locus(f\,|_S) \subset S.$$

For any real positive r we denote

$$Ball(r) = \left\{ v \in \mathbb{Z}^d : \|v\| \le r \right\}.$$

Then we define a sequence of functions

$$g_i = f\,|_{Ball(i)} \quad \text{for} \quad i = 1, 2, 3, \ldots$$

and a sequence of functions f_i, where $f_1 = g_1$ and $f_i = g_i - g_{i-1}$ for $i = 2, 3, 4, \ldots$ Let us prove our conditions for thus defined f_i. First, it follows from (27) that

$$\sum_{v \in \mathbb{Z}^d} impact(f\,|\,v) < \infty. \tag{28}$$

Further, since $g_n = f_1 + \cdots + f_n$, it is sufficient to estimate for arbitrary $x \in \Omega$

$$
\begin{aligned}
&|g_i(x) - f(x)| = \\
&|f\,|_{Ball(i)}(x) - f(x)| = \\
&|f(x\,|_{Ball(i)}) - f(x)| \le \\
&impact(f\,|\,\mathbb{Z}^d \setminus Ball(i)) \le \\
&\sum_{v \in \mathbb{Z}^d \setminus Ball(i)} impact(f\,|\,v),
\end{aligned}
$$

which tends to zero when $i \to \infty$ due to (28) . Thus (4) is proved.

Now let us prove (5) . First let us estimate

$$
\begin{aligned}
&\sup |f_i| \le \sup |g_i - g_{i-1}| = \\
&\sup \left| f\,|_{Ball(i)} - f\,|_{Ball(i-1)} \right| = \\
&\sup \left| f(x\,|_{Ball(i)}) - f(x\,|_{Ball(i-1)}) \right| \le \\
&\sum_{i-1 < \|v\| \le i} impact(f\,|\,v).
\end{aligned}
\tag{29}
$$

Due to (27) , every term of the sum (29) does not exceed $const \cdot i^{-(2d+1)}$. The number of terms in the sum (29) does not exceed $const \cdot i^{d-1}$. Therefore the sum (29) does not exceed $const \cdot i^{-(d+2)}$. Thus $\sup |f_i| \le const \cdot i^{-(d+2)}$. Therefore the sum (5) converges.

Now notice that $locus(g_i) \subset Ball(i)$, whence $locus(f_i) \subset Ball(i)$ for all i . Hence follows (6) with $L = 1$. *Proposition 5 is proved.*

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