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Gács, Peter (1-BOST-C)
Reliable cellular automata with self-organization. (English summary)
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A reader's guide to P. Gács's "positive rates" paper: "Reliable cellular automata with self-organization" [J. Statist. Phys. 103 (2001), no. 1-2, 45–267; MR1828729 (2002c:82058a)]. (English summary)
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FEATURED REVIEW.two papers will be called Gács's article and Gray's article. Gács's article is a contribution to two branches of science at once: to the theory of reliable systems made of unreliable components and to the theory of phase transitions in statistical physics. Gray's article explains Gács's results, which is very welcome because Gács's article is difficult to read.

For a long time it was a common opinion in statistical physics that phase transitions and critical phenomena can occur only in systems whose dimension is greater than one. For example, §152 of L. D. Landau and E. M. Lifshits's [*Course of theoretical physics. Vol. 5*, Translated from the Russian by J. B. Sykes and M. J. Kearsley. Second revised and enlarged edition, Pergamon, Oxford, 1968; MR0242449 (39 #3780)] was called "The impossibility of the existence of phases in one-dimensional systems", and an argument of physical nature was presented in support of this impossibility. However, these ideas were formed in dealing with models of equilibrium, which had no time parameter (e.g. the Ising model).

In the 1960s a theory of random processes with local interaction started to develop rapidly. These systems' components are usually placed in $\mathbf{Z}^d$, where d is called the dimension of the system, and interact with their nearest neighbors in discrete or continuous time. Should the time variable be counted as an additional dimension? In other words, can one-dimensional processes with local interaction display critical behavior?

This question was discussed at Moscow seminars. First, Dobrushin predicted that there should be simple models with $d = 1$ showing some analog of phase transitions. Then Piatetski-Shapiro organized computer modelling which suggested that systems with $d = 2$ displayed phase phenomena while systems with $d = 1$ did not [N. B. Vasil′ev, M. B. Petrovskaya and I. I. Piatetski-Shapiro, Automat. Remote Control **1969**, 1639–1642 (1970); Zbl 0205.17803].

After that the "positive rates conjecture" was advanced by several people claiming that this is the case with all systems of this sort. Gray's article describes this and the further work, with only one important omission: he does not mention B. S. Tsirelson's work [in *Locally interacting systems and their application in biology (Proc. School-Sem. Markov Interaction Processes in Biology, Pushchino, 1976)*, 15–30. Lecture Notes in Math., 653, Springer, Berlin, 1978; MR0504458 (58 #20899)], which was quite rigorous (unlike Kurdyumov's) and in some sense refuted the positive rates conjecture, except that his construction was essentially non-homogeneous both in space and in time. However, some important ideas were already present there, especially the idea of an infinite hierarchy (mentioned in the abstract of Gács's article).

Here is one way to formulate the positive rates conjecture. Let us first define a deterministic cellular automaton. Given a finite set S and a function $\mathrm{Tr}\colon S^3 \to S$, for any initial configuration $\eta(s, 0)$ we define the corresponding trajectory $\eta(s, t)$ for all $t > 0$ inductively:

$$\eta(s, t) = \mathrm{Tr}(\eta(s-1, t-1), \eta(s, t-1), \eta(s+1, t-1)).$$

(Here $s \in \mathbf{Z}$, so $d = 1$. For $d > 1$ the definition is analogous.) Let us say that a configuration η has an error at a point (s, t) if this equality does not hold. For any $\varepsilon > 0$ we can define a class of probability distributions on the set of configurations whose restriction to the initial layer $t = 0$ is concentrated in the initial configuration and for which errors are rare enough, for example, for any k points the probability that there are errors at all of them does not exceed ε^k. Let us call this system fault-tolerant if it admits a stable trajectory, where a trajectory η is called stable if the supremum of the probability that the (s, t)-th component is different from $\eta(s, t)$ over all s, t and distributions in our class tends to zero when $\varepsilon \to 0$. Existence of stable trajectories for all $d > 1$ was first shown by Toom for S containing only two elements, ε of the order $0.1 \sim 0.001$ and simple transition functions, one of which was proposed by Vasil′ev, Petrovskaya and Piateski-Shapiro [op. cit.] and is known as NEC or Toom voting. However, this method fails in the one-dimensional case. One version of the positive rates conjecture is that there are no stable trajectories in the one-dimensional case.

Gács's article is as of now the most sophisticated contribution to the theory of cellular automata. It not only refutes the positive rates conjecture (Gács has done this before) but develops a method which makes it possible to obtain several other unexpected results. Namely, Gács constructs a model which (here we follow Gray's article): (i) is capable of imitating in a fault-tolerant manner arbitrarily complex behaviors (because one-dimensional deterministic cellular automata can implement all algorithms); (ii) can be implemented in continuous time; this is not especially unexpected, but is technically difficult and expands the area of its applicability; (iii) can be made "self-organizing"; this property may be interesting for biologists; (iv) has the "positive capacity property"; it can reliably store one bit of information per site in the presence of noise; this promises rich opportunities for computer science.

Gács's main result is Theorem 6.15 on page 118. We cannot present it here because it uses notions whose definitions occupy practically all of the preceding text. Informally speaking, it claims for the continuous time the same as Theorem 5.8 on page 104 claims for the discrete time: that practically any deterministic cellular automaton can be reliably modelled by a random cellular automaton provided the latter has enough states and a small enough frequency of errors. (Gács has to use an S containing something like 2^{100} elements and ε of the order of 2^{-50}.)

Gács's method works also for finite systems. Essentially, he shows that there exists a finite-state machine M such that for any computer program π and any desired level of reliability $p < 1$, the program π can be run with reliability p on a sufficiently large one-dimensional array of copies of M, each one communicating only with its nearest neighbors, even if all these machines have a small enough, but constant, positive error rate.

Gács's articles are technically very complicated and resemble computer programs, so it is only natural that they contain bugs which are difficult to find all at once: they are found little by little. (One important mistake in a previous version was found by Robert Solovay.) However, it is important to notice that all these bugs have been corrected, which means that Gács's ideas are basically sound. There were doubts whether Gács's work was correct at all, so Gray has to state explicitly that it is, although even the present version contains local mistakes. (For example, there is an omission in Theorem 5.8 which is important when N is finite: it must be required that $K \leq K(N)$, where $K(N)$ is defined in (4.4).) Essentially, Gács's and Gray's articles taken together constitute the conspectus of a book, which needs to be written.

Reviewed by *Andrei Toom*