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A FAMILY OF UNIFORM NETS OF FORMAL NEURONS

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I. Introduction. In this paper we study Markov chains describing the behavior of some nets of formal neurons which possess the property of spontaneous activity. The problem of studying these chains was proposed in [1]. We introduce the required definitions. We describe the family of Markov chains, which we call M_n (n is a natural number or ∞). The state of such a Markov chain is given by prescribing the values of the variables a_i which can be equal to 0 and 1. The subscript i runs through the following values:

$$i = 1, 2, \cdots, n \quad \text{for natural } n$$
$$i = 0, \pm 1, \pm 2, \cdots, \quad \text{if } n = \infty.$$

Every variable a_i depends on the time $t = 0, 1, \cdots$; the state a_i at the moment t will be denoted by a_i^t. The transition from t to $t+1$ is defined in the same way for all chains. For $t > 0$ we introduce the variables b_i^t, interpreted in [1] as spontaneous self excitation. Each of them equals one with a probability independent of the other b_i^t. We put by definition for $t > 0$:

$$a_i^t = \begin{cases} 1, \text{ if at least one of the following two events is satisfied:} \\ \qquad a_{i-1}^{t-1} = a_i^{t-1} = 1 \text{ or } b_i^t = 1 \\ \qquad \text{where for natural } n \ a_0 \equiv a_n; \\ 0 \text{ otherwise.} \end{cases} \tag{1}$$

We assume the following initial conditions:

$$\text{for } t = 0 \text{ all } a_i^0 = 0. \tag{2}$$

Every Markov chain M_n (n finite) is obviously finite and has 2^n states while the state in which $a_1 = a_2 = \cdots = a_n = 1$ is absorbing. The set of states of the chain M_∞ has the cardinality of the continuum; by the transition rules (1) and by the initial conditions there is defined a measure on this set for every $t > 0$, i.e. for an arbitrary finite number m of subscripts i and an arbitrary choice of constants c_i ($i \in m$), equal to zero or one, there is defined the probability of the event given by: $a_i^t = c_i$ for all $i \in m$.

We denote by $P_{r,n}^t(\theta)$ the probability that $a_i^t = a_{i+1}^t = \cdots = a_{i+r-1}^t = 1$ in the chain M_n. It is easy to show that $P_{r,n}^t(\theta)$ is monotone increasing in θ. For the initial conditions under consideration it is shown in [1] that there exists the limit

$$\lim_{t\to\infty} P_{r,\infty}^t(\theta) = P_{r,\infty}^\infty(\theta).$$

In [1] estimates from below are obtained for the probabilities $P_{r,n}^t(\theta)$, from which it follows that for sufficiently large θ (for instance, $\theta \geq 0.37$):

a) for $n = \infty$ and arbitrary r, $P_{r,\infty}^\infty = 1$; (3)

b) for natural n the mathematical expectation $T_n(\theta)$ of the first time t for which $a_i^t = a_2^t = \cdots = a_n^t = 1$, does not exceed

$$\text{const}\cdot\log n. \tag{4}$$

On the basis of simulating these nets on the ÈVM the authors of [1] assume that the assertions (3), (4) are satisfied only for the values $\theta > \theta_0 > 0$ $(\theta_0 \approx 0.3)$, while for $\theta < \theta_0$ we have

a) for $n = \infty$, $P_{r,\infty}(\theta) < 1$; (5)

b) for natural n, $T_n(\theta) \geq C^n$ $(C = C(\theta) > 1)$. (6)

For $n = \infty$ this problem has been studied in [2], where assertion (5) has been proved for sufficiently small values of $\theta > 0$ (no estimate for θ is given in the paper). In the present paper we obtain estimates from above for the probabilities $P^t_{r,n}(\theta)$, from which follow the assertions (5), (6) for sufficiently small θ (for instance $\theta \leq 0.07$).

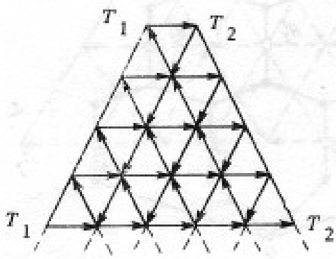

Figure 1

More precisely, the assertion (5) is proved for $\theta < 1/C$, where the constant C has the simple geometric meaning:

$$C = \lim_{k\to\infty} \sqrt[k]{N^k}; \tag{7}$$

N^k being the number of ways by which it is possible to pass in Figure 1 from $T_1 T_1$ to $T_2 T_2$, moving only in the direction of the arrows, not passing through one point more than once and making exactly k steps to the right while the steps to the left upwards and to the left downwards cannot immediately follow one another. The existence of the limit (7) follows from the inequality $N^{k+l} \leq N^k N^l$.

A decisive step in these proofs is the interpretation of the probabilities $P^t_{r,n}(\theta)$ as probabilities of the occurrence of some contact schemes of unreliable elements similar to those considered in [3]. The fact that the function $P^{\infty}_{r,\infty}(\theta)$ is smaller than 1 for $\theta < \theta_0$ (for small θ it behaves like θ^r) and equal to one for $\theta > \theta_0$ (see the graph of $P^{\infty}_{1,\infty}$ in [1]) is presented as a corollary of the fact that these schemes possess properties similar to the property of reliability described in [3], while their width for $t \to \infty$ tends to ∞, and their length remains equal to r.

II. As in [3] a contact scheme means a graph whose edges can conduct a current. We shall, however, consider the edges one-sided, i. e. capable of conduction in one direction and not conducting in another direction.

The scheme conducts from the pole M to the pole N if there exists a conducting path from M to N, i. e. a sequence of edges $MA_1, A_1A_2, \cdots, A_nN$, where MA_1 conducts from M to A_1, etc.

The scheme is called planar if its graph is planar. For a planar scheme S with one-sided edges there is defined a dual scheme $\bar{S}$. Its graph is dual to the graph of the scheme S, and the edge KL of the scheme $\bar{S}$, corresponding to the edge AB of the scheme S, where K is the left and L the right domain adjacent to the edge AB of the planar graph of the scheme S, conducts from K to L if AB does not conduct from A to B, and conducts from L to K if AB does not conduct from B to A.

The scheme S does not conduct from M to N if and only if there is in the scheme $\bar{S}$ a closed conducting path $K_1K_2\cdots K_n$ (i.e. K_1K_2 conducts from K_1 to K_2 etc.), separating M from N and going clockwise around M.

III. Estimate of $P^t_{1,\infty}(\theta)$. We now return to our Markov chains. In Figure 2 we denote by small circles the points A^u_j. In every point A^u_j we place a variable a^0_j for $u = 0$ and b^u_j for $u > 0$. Then $a^t_i = 0$ if and only if from the point A^t_i we can reach the sequence of points A^0_j by going downwards only through points A^u_j occupied by zeros. This graphic representation was noticed immediately after paper [1] had been written. For instance, L. G. Mitjuščin showed with its aid that the probability

$P^t_{1,\infty}(\theta)$ exactly equals the probability that in the chain M at time t all a^t_i are equal to 1 given that at $t = 0$ one of them was equal to zero and the remaining ones equal to one.

We now construct a planar scheme $S^t_{1,\infty}$, which conducts from M to N if and only if $a^t_i = 0$. For $t = 3$ the scheme $S^t_{1,\infty}$ is represented in Figure 2. Its vertices are represented by black points. It is constructed in the following manner: through every point A^u_j, $i - t + u \leq j \leq i$, we draw a vertical edge, denoted in the sequel also by A^u_j, which conducts upwards if and only if the corresponding variable which a^t_i depends on, i.e. a^u_j for $u = 0$ and b^u_j for $u > 0$, equals zero. The lower end of such an edge A^u_j for $u > 0$ is joined by the inclined edges with the upper ends of the edges A^{u-1}_{j-1} and A^{u-1}_j. These inclined edges always conduct upwards, i.e. in the direction of the arrows and never in the converse direction. As pole N we take the upper end of the edge A^t_i. The pole M is joined to the lower ends of the edges A^0_j by edges always conducting upwards (i.e. from M to these ends). The conductivity of these edges, as well as of the edges A^u_j, downwards is immaterial. It is convenient to consider that these edges always conduct downwards.

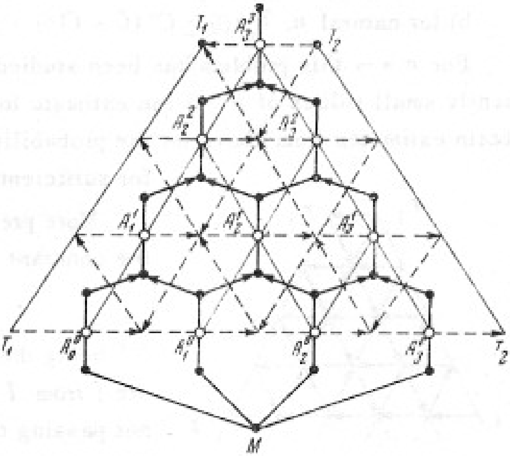

Figure 2

We now construct the scheme $\bar{S}^t_{1,\infty}$, dual to $S^t_{1,\infty}$. It is represented in Figure 2. The edges forming the lines $T_1 T_1$ and $T_2 T_2$ always conduct to both sides and are joined by an always conducting edge, which is not drawn. The remaining edges of the scheme $\bar{S}^t_{1,\infty}$ are given by the broken lines. Among those the edges with the arrows (the inclined ones) obviously always conduct in the direction of the arrows and never in the converse direction. The horizontal edges B^t_i, corresponding to A^t_i, conduct to the right with probability θ for $t > 0$ and do not conduct for $t = 0$, and never conduct to the left. We shall omit the undrawn edge joining T_1 and T_2 and denote $\bar{S}^t_{1,\infty}$ without this edge by $\bar{S}'^t_{1,\infty}$.

The probability $h_{T_1,T_2}(\theta)$ of the event that $\bar{S}'^t_{1,\infty}$ conducts from T_1 to T_2 equals $P^t_{1,\infty}(\theta)$. We estimate it from above. This probability is smaller than the sum of probabilities of all the events of the type: "on a given path from T_1 to T_2 all the edges conduct." The probability of such an event equals θ^k, where k is the number of horizontal edges on that path. We can consider only paths which pass no vertex more than once and on which inclining edges of different directions do not succeed one another. The number of such paths, containing k horizontal edges, exactly coincides with N^k in (7). Thus:

$$h_{T_1T_2}(\theta) < \sum_{k=1}^{\infty} N^k \theta^k. \tag{8}$$

N^k can easily be estimated from above by the number 3^{3k}, if we code every path by the sequence of digits 1, 2, 3, where the digit 1 denotes an edge going downwards to the left, 2 to the right and 3 upwards to the left. Then

$$h_{T_1T_2}(\theta) < \sum_{k=1}^{\infty} N^k \theta^k \leqslant \sum_{k=1}^{\infty} 3^{3k} \theta^k = \sum_{k=1}^{\infty} (27\theta)^k = \frac{27\theta}{1 - 27\theta}.$$

For $\theta < 1/54$ this sum is smaller than one. We have thus proved the assertion (5) for $\theta < 1/54$.

The estimate of N^k can be somewhat improved if we note that in the sequence codifying the path the digits 1 and 3 cannot stand next to each other. Then we obtain $N^k \leq \text{const} \cdot (14.1)^k$.

Analogously we can take into consideration some other restrictions, such as the fact that sequences codifying the paths cannot contain the combinations 321 and 123, etc.

IV. Other estimates. First we estimate $P^t_{r,\infty}(\theta)$. For this purpose we construct the scheme $S^t_{r,\infty}$, which conducts from M to N if and only if at least one of these variables equals zero. This scheme can be obtained from the scheme $A^{t+r-1}_{1,\infty}$ if we omit from this scheme all the edges A^u_j for $u > t$ and the edges with the arrows joining them, while we join the upper ends of the edges A^t_j with the vertex N. For this scheme we construct a dual scheme with a rejected edge $\overline{S}'^t_{r,\infty}$ and in a way analogous to the preceding one we estimate from above the probability $h_{T_1 T_2}(\theta)$ of the event that the scheme $\overline{S}'^t_{r,\infty}$ is conducting:

$$h_{T_1T_2}(\theta) \leqslant \sum_{k=r}^{\infty} N^k\theta^k \leqslant \text{const} \sum_{k=r}^{\infty} (C\theta)^k = \text{const}\, \frac{(C\theta)^r}{1-C\theta},$$

where C is defined in (7).

For any $\theta < 1/C$ we can find an r for which the above expression is smaller than one. If we utilize the inequality

$$1 - P^t_{r,\infty}(\theta) \leqslant r\,[1 - P^t_{1,\infty}(\theta)],$$

we may deduce that for any $\theta < 1/C$ the probability $P^t_{1,\infty}(\theta)$ also does not tend to one.

For finite n it is easy to construct schemes $S^t_{r,n}$, analogous to $S^t_{r,\infty}$ and schemes $\overline{S}^t_{r,n}$ dual to them. We confine ourselves to the case $r = n$. It is easy to obtain an estimate, for $\theta < 1/8$:

$$P^t_{n,n}(\theta) \leqslant t \sum_{k=0}^{\infty} (2\sqrt{2})^{n+4k}\theta^{n+2k} = t\,\frac{(2\sqrt{2}\theta)^n}{1-8\theta}.$$

Hence the mathematical expectation $T_n(\theta)$ of the first time for which all $a^t_i = 1$, is not smaller than $(1-8\theta)/2(2\sqrt{2\theta})^n$.

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