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Particle systems with variable length*

André Toom

Abstract. We present a new class of one-dimensional particle systems, in which the number of components may change in the process of interaction. We suggest that some of these systems display spontaneous symmetry breaking.

Keywords: 1-D particle systems, local interaction, spontaneous symmetry breaking, positive rates conjecture.

Mathematical subject classification: Primary 60K35; Secondary 60G35, 82C22.

The bulk of modern theory of interacting particle systems as presented e.g. in [9] (continuous time) or [1] (discrete time) is based on the assumption that the set of components, also called the space, does not change in the process of interaction; usually it is $\mathbb{Z}^d$ or $\mathbb{R}^d$, where d is dimension. Elements of this space, also called sites, may be in different states (e.g. 1 and 0, often interpreted as presence vs. absence of a particle), but the sites themselves do not appear or disappear in the process of functioning. This assumption is not the only possible one and seems to be motivated partially by mathematical convenience. Here we present another approach.

1 Systems with variable length: general approach

The first purpose of this article is to present a new class of one-dimensional locally interacting particle systems, which we call "systems with variable length". In this paper we concentrate only on the one-dimensional case assuming that every component has only two states: 0 and 1.

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By *words* we mean finite sequences of zeros and ones. By length of a word we mean the number of letters (zeros or ones) in this word. The length of any word W is denoted $|W|$. By definition, there is one word with length zero: the empty word denoted Λ. We can write the set of words as

$$L = \bigcup_{k=0}^{\infty} \{0, 1\}^k,$$

where $\{0, 1\}^k$ is the set of words with length k. In the finite case the set L will serve as the set of states of our process, which are called configurations in this case. In the infinite case the set of configurations is $\{0, 1\}^{\mathbb{Z}}$, the set of bi-infinite sequences of zeros and ones.

Our main innovation is more easy to explain in the finite case: the number of components, which is finite all the time, may change in the process of interaction because, when one combination of states is substituted by another, the lengths of these combinations may differ from each other. We shall write a generic substitution rule in the form

$$old \xrightarrow{r} new. \tag{1}$$

Informally speaking, this means that whenever the word *old* is met in the configuration, it is substituted by the word *new* with a rate r. We shall say that a substitution (1) has *constant length* if $|old| = |new|$ and *variable length* otherwise.

Let us mention several simple substitutions. In fact, in every case we present two symmetric substitution with a common name, but which may be used with different rates. In every case j is an index of a component.

Conversion: $0 \to 1$ and $1 \to 0$. The j-th component changes its state from 0 to 1 or from 1 to 0. The number of components does not change, so this is a substitution with constant length. All the next substitutions have variable length.

Birth: $\Lambda \to 0$ and $\Lambda \to 1$. A new component in the state 0 or 1 appears between the $(j-1)$-th and the j-th component. The number of components increases by one.

Death: $0 \to \Lambda$ and $1 \to \Lambda$. The j-th component disappears if it is 0 or if it is 1, the $(j-1)$-th and $(j+1)$-th components become neighbors and the number of components decreases by one.

Mitosis: $0 \to 00$ and $1 \to 11$. The j-th component duplicates if it is 0 or if it is 1. The number of components increases by one.

Annihilation: $01 \to \Lambda$ and $10 \to \Lambda$. If $x_j = x_{j+1}$, nothing changes, if $x_j < x_{j+1}$ or if $x_j > x_{j+1}$, both components disappear, in result of which the number of components decreases by two and the $(j-1)$-th and $(j+2)$-th components become neighbors.

Although systems with variable length seem to have never been studied in general, they may be useful in modeling various natural phenomena, in which we are dealing with long chains of interacting units, whose number may change. For example, some biological structures are long and thin and therefore may be approximated by one-dimensional models, in which components may represent cells or microorganisms, which may divide or die or be infected or otherwise influenced. In linguistics, we may study evolution of utterances, in the course of which the number of units (syllables, phonems, letters) may change also. For example, the Portuguese word "geral" comes from "general" by omission of the middle syllable. Computer modeling of systems with variable length is also handy: if we represent the system as a doubly linked list, it is easy to write a soubroutine for any local substitution.

We consider systems, functioning of which is determined by a finite list of substitutions

$$old_i \xrightarrow{r_i} new_i, \quad i = 1, \ldots, n. \tag{2}$$

What happens to a configuration x in a small time Δt, can be informally described as follows. We prepare n types of marks and for every type $i = 1, \ldots, n$ we mark a fraction $r_i \cdot \Delta t$ of indices existing at time t with the i-th mark at random. Since $\Delta t \to 0$, we may assume that all marks are far enough from each other, so that the following instruction is unambiguous: For every index j marked with the i-th mark we check whether the word, formed by $|old_i|$ components starting with the component x_j, coincides with old_i. If it does, we eliminate these components and substitute them by new_i, otherwise we change nothing. In particular, if $new_i = \Lambda$, we simply eliminate old_i. On the other hand, if $old_i = \Lambda$, we simply insert new_i between the $(j-1)$-th and j-th components of the present configuration.

In the finite case our process is a Markov chain with a countable set L of states and continuous time. Let us write down the rates of this Markov chain. For any finite sequence of words $W_1, \ldots, W_n$ we call *concatenation* and denote concat$(W_1, \ldots, W_n)$ the word obtained by writing all the words in the brackets one after another in that order in which they are listed. The rate of going from a state x to any other state y equals

$$p(x \to y) = \sum_{j=1}^{|x|-|old_i|} \sum_{i=1}^{n} \pi_{i,j}(x \to y), \tag{3}$$

where

$$\pi_{i,j}(x \to y) = \begin{cases} r_i & \text{if there exist words } W_0 \text{ and } W_1 \text{ such that } |W_0| = j-1 \\ & \text{and} \quad x = \text{concat}(W_0,\ old_i,\ W_1) \\ & \text{and} \quad y = \text{concat}(W_0,\ new_i,\ W_1), \\ 0 & \text{otherwise.} \end{cases}$$

Generally we define any sum with an empty set of summands to be equal to zero. In particular, if $|x| - |old_i| < 1$, the right part of (3) equals zero.

Thus the process in the finite case is defined. In the infinite case we can define our process as a limit of the finite case, but we shall do it in general in another publication. Here we shall do it only for a special case.

2 Spontaneous symmetry breaking

One may suggest that systems with variable length are essentially the same as traditional systems, only more cumbersome. We think otherwise. Let us concentrate on one problem — the possibility of phases in 1-D systems.

For a long time it was a common opinion among physicists that phase transitions are impossible in one-dimensional systems. For example, §152 of Landau and Lifshitz's "Statistical Physics" was called "The impossibility of the existence of phases in one-dimensional systems" and an argument of physical nature was presented in support of this impossibility. Another example: "In one dimension bosons do not condense, electrons do not superconduct, ferromagnets do not magnetize, and liquids do not freeze" [10], p. vi. Based on this tradition and some computer simulations [11], several authors (see Chapter 4, section 3 of [9], or p. 115 of [1] or [6] or [13]) proposed a "positive rates conjecture", which informally may be expressed as follows: "all one-dimensional particle systems with non-degenerate local interaction are ergodic, that is cannot display analogs of phase transitions." When this conjecture was formulated, various people tried to produce counter-examples to it [14, 4, 7, 2, 12], but none of them succeeded to refute this hypothesis in the proper sense for one or another reason.

Only after fifteen years of hard work Gács [3] developed a very involved system, which completely refuted the positive rates conjecture. However, this system has an enormous number ($\approx 2^{100}$) of states of a single component, very complicated deterministic rule of interaction and a very small ($\approx 2^{-50}$) probability of deviation from this rule. Gray [5], who has studied the question in detail, believes that no much simpler system can refute the conjecture. So much about the traditional 1-D particle systems.

It seems that systems with variable length provide new, unexpected possibilities. Our main example is pretty simple, every component of it has only two possible states 0 and 1, the interaction is local, the rules of interaction are symmetric and the rate ε with which any 0 turns into 1 and vice versa is positive. Nevertheless, computer simulation and some approximations suggest that most alive components remain in the state 0 and vice versa for all $t \geq 0$ provided the initial condition was "all zeros" and ε is small enough.

This example makes a contrast with the positive rates conjecture, but does not refute it, because all those who proposed that conjecture meant systems with constant length.

Main example. *The system with these three substitutions:*

$$0 \xrightarrow{\varepsilon} 1, \quad 1 \xrightarrow{\varepsilon} 0, \quad 01 \xrightarrow{1} \Lambda, \quad 10 \xrightarrow{1} \Lambda. \tag{4}$$

Let us explain how to define the infinite process in this case. We should be careful with indices. Initially our components are indexed using integer numbers: a finite range of them in the finite case or all of them in the infinite case. However, in the process of functioning some indices disappear due to annihilations. Let us denote I_t the set of indices at time t. If $s \in I_t$, we call the point $(s, t) \in \mathbb{R}^2$ *alive*, otherwise we call it *dead.*

Our process can be interpreted as a process in which the set of sites remains $\mathbb{Z}$ all the time and every site has three possible states 0, 1 or *dead* with a special non-local rule of interaction in which alive sites interact as if the dead sites were absent.

When we speak about distributions on the segment $[a, b]$, we mean distributions on $\{0, 1, dead\}^{[a,b]}$ and when we speak about convergence of these distributions, we mean convergence on every element of $\{0, 1, dead\}^{[a,b]}$.

Theorem 1. *Take the system (4) and start with a finite configuration "all zeros" of length $2w+1$ indexed by integer numbers in the range $[-w, \ldots, w]$. Then for any $t > 0$ and any integer $r > 0$ the restriction of the distribution of our system at time t to the segment $[-r, r]$ has a limit when $w \to \infty$ and this limit is invariant under space translations.*

We shall prove this theorem in another publication. Let us assume that it is proved and take this limit as a definition of the infinite system.

Conjecture. *Take the infinite system (4) with a bi-infinite sequence of zeros as the initial condition. Then the density of ones among alive sites, that is the conditional probability*

$$Prob\left(x(0,t) = 1 \,\middle|\, (0,t) \textit{ is alive}\right) \tag{5}$$

never exceeds a function of ε, which tends to zero when $\varepsilon \to 0$.

Let us emphasize that our hypothesis, even if proved, is useless as such because all the finite systems in this example shrink on the average pretty quickly, thereby losing any non-symmetry. What we need is a similar statement for systems similar to (4), but with addition of symmetric births and/or mitosis, due to which the finite systems would grow.

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André Toom
Federal University of Pernambuco
Department of Statistics
50740-540 Recife, PE
BRAZIL
E-mail: toom@cox.de.ufpe.br, toom@member.ams.org