REVIEWS

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Mathematical Circles (Russian Experience) by Dmitri Fomin, Sergey Genkin, and Ilia Itenberg. Translated from Russian by Mark Saul. American Mathematical Society, 1996, 272 pp, $29.00

Reviewed by **Andre Toom**

I shall call this book *Circles* for short. *Circles* is written by three Russian mathematicians with whom I have much in common. For many years we participated in similar activities including organization of mathematical competitions and teaching informal classes called "circles" in Russia. Six years ago I moved to the United States and worked at several universities. Based on this experience I want to reflect on how *Circles* can be used in America.

First of all, it is a rich collection of good problems. In addition there are useful notes for teachers. I think that Appendix A, which describes several types of mathematical contests, will be especially interesting for those who are dealing with all forms of cooperative learning. To get some taste of the book, let us look at a few problems.

> Problem 1 on page 1
> *A number of bacteria are placed in a glass. One second later each bacterium divides in two, the next second each of the resulting bacteria divides in two again, et cetera. After one minute the glass is full. When was the glass half-full?*

Some students propose a half-minute as the answer, implicitly assuming that the growth is linear. This problem shows in a dramatic way how different is exponential growth from linear.

> Problem 10 on page 172
> *An evil king wrote three secret two-digit numbers a, b, and c. A handsome prince must name three numbers X, Y, and Z, after which the king will tell him the sum $aX + bY + cZ$. The prince must then name all three of the King's numbers. Otherwise he will be executed. Help him out of this dangerous situation.*

It may seem at first that it is impossible to determine three variables from one equation and that the prince is doomed, but in fact he can save himself. The solution is based on a fruitful idea that can be used to introduce the students into the basics of information theory.

Circles may be very useful wherever there are classes devoted to solving non-standard problems. There may be more such classes than we are aware of at some selective schools. But Russian circles were not selective in any formal sense; in fact, anybody might drop in. So I did about forty years ago: I simply took a trolleybus, went to the old university campus in downtown Moscow and started to attend informal classes taught by students of Moscow University. I submitted no formal application, paid nothing, and got no grades, but there I became a professional mathematician. Most classes discussed olympiad-style problems, but our teacher, Sasha Olevskii, was keen on epsilon-delta arguments of mathematical analysis, and for me it was just the right thing. Later I taught circles for decades, and now I meet my former students at various places, including Boston and Tel Aviv.

However, I think that the possible use of *Circles* is wider. I cannot imagine mathematical education without teaching students to solve mathematical problems. I am astonished when educators discuss some special "problem-solving approach" to teaching mathematics as if the opposite approach ever made sense. Universities offer special "problem-solving" courses. Does this mean that other courses do not teach to solve problems? Anyway, "problem-solving" courses should be welcomed, and *Circles* deserves a very prominent place among candidates for textbooks for such courses. In addition, many universities offer courses in mathematics without specifying exactly what these courses are about. When I teach such courses, I try to teach my students to solve as many problems as possible. It was very difficult for me to find a textbook for that. Now I have found at least one: *Circles.*

At this point I must admit some inconsistency. I have to agree with the translator's disclaimer: "This is not a textbook." At the same time I am going to use *Circles* as a textbook. The point is that it is an unusual textbook. It is unusual in several respects, but let us concentrate on the following. First, most problems in *Circles* need a rigorous approach. Some of them explicitly require proofs, while others ask questions whose answers involve argumentation. Second, most problems in most textbooks can be solved in a way that is explained in advance. Many such problems have identical mathematical structure presented in different "real world" guises. All this is absent in *Circles*. Most problems need a new idea although problems are grouped and some general theory is explained.

Now we approach a very important notion: *transfer of training*. Let us imagine the set of all possible problems in a branch of mathematics as a metric space. Every particular problem is a point in this space and similar problems are close to each other. By discussing a problem with our students, we cover a sphere with the center at this problem and radius equal to our students' ability to transfer their training from this problem to similar problems. Our purpose is to cover the greatest possible space with these spheres. Is transfer of training possible? For the authors of *Circles*, for me, and for all who have ever taught in this style the answer is evident: "yes, of course, transfer of training is possible and it is closely related to another precious human ability: generalization."

But some American educators have questioned the importance or the very possibility of transfer of training. Instead they proposed to solve only such problems in class that people face in everyday life. These educators have sent many messages to the effect that students are not interested in problems that have no immediate real-world use. As a result, some students' ability for generalization and transfer of training is almost completely atrophied. As soon as a problem on a test slightly deviates from problems solved in advance, they complain, "We did not solve such problems." Whichever course these students take, they learn to solve only those particular problems that the teacher chose to explain. Their spheres have radii that are close to zero and their total measure is also close to zero. It is a safe bet that the problems they have to solve after graduation practically never coincide with those few they have learned to solve in class.

All this does not mean that the manner in which problems are formulated is unimportant. It is very important and we may reproach *Circles* for some neglect in this respect. Let us illustrate these ideas by some examples.

Problem 11 on page 53
A woodman's hut is in the interior of a peninsula which has the form of an acute angle. The woodman must leave his hut, walk to one shore of the peninsula, then to the other shore, then return home. How should he choose the shortest such path?

This is a very useful problem. It introduces students to the important realm of geometrical transformations. It also gives the teacher a chance to speak about relations between mathematics and physics because light also "chooses" the shortest path. Regrettably, no motivation is given why the woodman should walk in such a strange way. In Steinhaus's book a similar problem is formulated more vividly: "An Arab wishes to return to his tent, but on the way he wants to feed his horse and draw water from a river." [2, pp. 111–112]. Note that my criticism has nothing to do with the silly requirements of straightforward "real-world" relevance. The real relevance is through theory, as usual. Another example: in problem 22 on page 3 there is a strange river that makes a 90° turn. In Kordemsky's book [1, problem 126 on p. 54] it was a moat surrounding a fort, which was more plausible and romantic.

Since I started to criticize *Circles*, let me continue to do it from my point of view as a teacher. *Circles* does not mention irrational numbers at all, not even the famous proof by contradiction that no square of a rational number equals 2. It is very easy to collect a series of problems in this vein. (This is what I do in my classes.) Why didn't the authors of *Circles* do it?

Some other important facts are present but not emphasized. For example, it takes attentive reading to find problem 53 on page 29, which is included into a section called "A few more problems" as if it were something optional: *Prove that there are infinitely many prime numbers*. Nothing is said about the importance of this fundamental fact.

On page 177 the authors explain one method to invent problems, which they call "inequalities à la Leningrad." A typical example, problem 1 on page 175: *Which number is greater*: 31^{11} *or* 17^{14}? Problems of this sort seem to have been a useful contribution to olympiads in the past, but I am afraid that the presence of calculators kills most of them. On the other hand, look at problem 43 on page 161: *If all the sides of a triangle are longer than* 1000 *inches, can its area be less than* 1 *square inch*? This problem and several similar ones were invented in the Moscow-based School by Correspondence, and these problems remain usable in presence of any technology.

Problem 10 on page 23 assumes acquaintance with divisibility tests for 3 and 9, which are introduced only in problem 31 on page 99. The same seems to be true for problem 76 on page 72; I could solve it only using divisibility tests. There is some confusion with problem 65 on page 71. The letter M is misplaced and K is absent in figure 123 on page 163.

I admit another inconsistency about students' age. *Circles* is addressed to 12–14 year old children. This contrasts with the way I am going to use *Circles*. All of my students are intelligent and motivated, but they are around twenty years old or more. Sometimes I wonder what have they been doing for so many years. Now they have to think about graduation and making a career. They are pressed by time and money concerns, some have to do odd jobs, some have children. Of course, all this interferes with their study. It would be much better for them to have started to solve non-trivial problems years ago.

This inconsistency is especially noticeable when one reads the initial Chapter Zero, intended for students of ages 10–11. "The problems of this chapter have virtually no mathematical content," naively claim the authors. Actually the problems of this chapter require the most fundamental ability—the ability for abstract thinking. This ability should by no means be taken for granted; it develops as a result of careful and well-thought schooling. Let us remember that most teachers of circles in Russia were not professional teachers. In most cases they were university students, inspired but unexperienced. Their teaching was successful due to the sound preparation provided by the national educational system. In my young years the Russian educational administration seemed very stupid to me, but now I see how efficient actually it was. Don't ask me how does this square with the tyrannical Soviet rule and the ailing Russian economy because I don't have all the answers.

Some people ask whether there are competent enthusiasts in America who could and would teach classes of creative problem solving. The answer is "yes, of course," but this is not the right question to ask. The right question is whether the educational system can teach the basics of mathematics so that children will be able to attend such classes.

REFERENCES

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2. H. Steinhaus, *Mathematical Snapshots*. New York, Oxford University Press, 1969.

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