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Velocities à la Galperin in Continuous Spaces

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Tese submetida ao Programa de Pós-Graduação em Matemática da Universidade Federal de Pernambuco como parte dos requisitos para obtenção do grau de **Doutor em Matemática**

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Co-orientador: Professor Alex Dias Ramos

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VELOCITIES À LA GALPERIN IN CONTINUOUS SPACES

Por

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L. H. S.

Solving problems is a practical skill like, let us say, swimming. We acquire any practical skill by imitation and practice. Trying to swim, you imitate what other people do with their hands and feet to keep their heads above water, and, finally you learn to swim by practicing swimming. Trying to solve problems, you have to observe and to imitate what other people do when solving problems and, finally, you learn to do problems by doing them.

— George Polya

How to solve it, 1945

Abstract of Thesis presented to UFPE as a partial fulfillment of the requirements for the degree of Doctor in Mathematics

VELOCITIES À LA GALPERIN IN CONTINUOUS SPACES

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Supervisor: Professor Andrei Toom
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UNLIKE Toom (1976) and Galperin (1975, 1977), we consider the *space* $\mathbb{R}^n$. The *set of states* is $M = \{0, 1, 2\}$. Maps $x : \mathbb{R}^n \to M$ are called $\mathbb{R}^n$-*configurations* and the set of all $\mathbb{R}^n$-configurations is denoted by $M^{\mathbb{R}^n}$. Consider a list $U = \{u_1, \dots, u_k\}$ of elements of $\mathbb{R}^n$ and $f : M^k \to M$ such that $a_1 \leq b_1, \dots, a_k \leq b_k$ implies $f(a_1, \dots, a_k) \leq f(b_1, \dots, b_k)$ and $f(a, \dots, a) = a$ for all $a \in M$. Any map $D : M^{\mathbb{R}^n} \to M^{\mathbb{R}^n}$ given by $(Dx)_p = f(x_{p+u_1}, \dots, x_{p+u_k})$ for all $p \in \mathbb{R}^n$ is called a *regular operator.*

For any $\mathbb{R}^n$-configuration x we denote $max(x) = \max\{\, x_p \,|\, p \in \mathbb{R}^n \,\}$. We say that D 2-*degrades* $x \in M^{\mathbb{R}^n}$ if there is $t_0 \in \mathbb{Z}_+$ such that $max(D^{t_0}x) < 2$. If D 2-degrades x, then we define the 2-*lifetime* of x under D as $\tau_2^D(x) = \min\{\, t \in \mathbb{Z}_+ \,|\, max(D^t x) < 2\}$. Otherwise, we say that the 2-lifetime of x is infinity and write $\tau_2^D(x) = \infty$. We call $x \in M^{\mathbb{R}^n}$ an *island* if the set $\{p \in \mathbb{R}^n \,|\, x_p \neq 0\}$ is bounded. An operator is called a 2-*degrader* if it 2-degrades all the islands.

Consider the island $d_R : \mathbb{R}^n \to M$, where $R \in \mathbb{R}_+$, given by $d_R(p) = 2$ whenever $\|p\| \leq R$ and otherwise $d_R(p) = 0$. We say that D is *linear 2-degrader* if there are $\lambda_D, \alpha_D > 0$ such that $\tau_2^D(d_R) \leq \lambda_D R + \alpha_D$ for all $R \in \mathbb{R}_+$.

Chapter 2 deals with the one-dimensional case, i.e., the continuous space $\mathbb{R}$. We have generalized the Galperin's $L_{0,2}$, $R_{0,2}$, $L_{2,0}$ and $R_{2,0}$. In special, we shall see that an one-dimensional 2-degrader is always linear 2-degrader. Chapter 3 presents a sufficient condition for a two-dimensional regular operator be linear 2-degrader. It is obtained by using one-dimensional tools introduced in Chapter 2. Chapter 4 presents a sufficient condition for a 2-degrader not be a linear 2-degrader one. Moreover, it also shows that under some assumptions over a regular operator there is a island that grows without bound.

Resumo da Tese apresentada à UFPE como parte dos requisitos necessários para a obtenção do grau de Doutor em Matemática

Velocidades à la Galperin em Espaços Contínuos

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Julho/2012

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Ao contrário de Toom (1976) e Galperin (1975, 1977), consideramos o *espaço* $\mathbb{R}^n$. O *conjunto de estados* é $M = \{0, 1, 2\}$. Aplicações $x : \mathbb{R}^n \to M$ são ditas $\mathbb{R}^n$*-configurações* e o conjunto de todas elas é denotado por $M^{\mathbb{R}^n}$. Considere uma lista $U = \{u_1, \ldots, u_k\}$ de elementos de $\mathbb{R}^n$ e $f : M^k \to M$ tal que $a_1 \leq b_1, \ldots, a_k \leq b_k$ implica $f(a_1, \ldots, a_k) \leq f(b_1, \ldots, b_k)$ e $f(a, \ldots, a) = a$ para todo $a \in M$. Qualquer aplicação $D : M^{\mathbb{R}^n} \to M^{\mathbb{R}^n}$ definido como $(Dx)_p = f(x_{p+u_1}, \ldots, x_{p+u_k})$ para todo $p \in \mathbb{R}^n$ é chamado um *operador regular*.

Dada uma configuração x, denotamos $max(x) = \max\{\, x_p \,|\, p \in \mathbb{R}^n \,\}$. Dizemos que D 2-*degrada* $x \in M^{\mathbb{R}^n}$ se existe $t_0 \in \mathbb{Z}_+$ tal que $max(D^{t_0}x) < 2$. Se D 2-degrada x, então definimos o 2-*tempo de vida* de x sob D como $\tau_2^D(x) = \min\{\, t \in \mathbb{Z}_+ \,|\, max(D^t x) < 2\}$. Caso contrário, dizemos que o 2-tempo de vida é infinito e escrevemos $\tau_2^D(x) = \infty$. Dizemos que $x \in M^{\mathbb{R}^n}$ é uma *ilha* se o conjunto $\{p \in \mathbb{R}^n \,|\, x_p \neq 0\}$ é limitado. Chamamos um operador de 2-*degrader* se este 2-degrada qualquer ilha.

Considere a ilha $d_R : \mathbb{R}^n \to M$, onde $R \in \mathbb{R}_+$, dado por $d_R(p) = 2$ quando $\|p\| \leq R$ e $d_R(p) = 0$ noutros casos. Dizemos que D é *2-degrader linear* se existem $\lambda_D, \alpha_D > 0$ tais que $\tau_2^D(d_R) \leq \lambda_D R + \alpha_D$ para todo $R \in \mathbb{R}_+$.

Capítulo 2 trata do caso uni-dimensional, i.e., $\mathbb{R}$. Lá, generalizamos $L_{0,2}$, $R_{0,2}$, $L_{2,0}$ e $R_{2,0}$ definidas por Galperin. Em especial, veremos que um 2-degrader uni-dimensional é sempre linear. Capítulo 3 apresenta uma condição suficiente para um operador regular bi-dimensional ser 2-degrader linear. Isto é obtido usando ferramentas uni-dimensionais introduzidas no Capítulo 2. Capítulo 4 apresenta uma condição suficiente para um 2-degrader não ser linear. Mais ainda, mostra que sobre certas condições sobre D existe uma ilha que cresce sem limites.

List of symbols

Symbol	Description	Page
M	the *set of states*, in most of the text $M = \{0, 1, 2\}$	11
x	a map $x : \mathbb{R}^n \to M$ is a $\mathbb{R}^n$-*configuration*	14
$M^{\mathbb{R}^n}$	the set of all $\mathbb{R}^n$-configurations	14
S^q	a *shift* by q	14
U	a finite subset of $\mathbb{R}^n$ called *neighborhood*	14
r_U	the *neighborhood radius* is $\max\{\,\lvert u_i\rvert \mid u_i \in U\}$	14
f	a map $f : M^k \to M$ called *transition map*	15
D	a *regular operator*	15
$\mathcal{L}(\,)$	the *left coordinate* of an (a, b)-ladder	19
$\mathcal{R}(\,)$	the *right coordinate* of an (a, b)-ladder	19
$l(\,)$	the *length* of an (a, b)-ladder, $l(\,) = \mathcal{R}(\,) - \mathcal{L}(\,)$	19
j_{ab}	an (ab)-*jump*	19
V_{01}	the (01)-*velocity*	19
V_{12}	the (12)-*velocity*	19
L_{02}	the *left* (02)-*velocity*	20
R_{02}	the *right* (02)-*velocity*	20
L_{20}	the *left* (20)-*velocity*	20
R_{20}	the *right* (20)-*velocity*	20
δ	an element $\delta \in \mathbb{R}^2$ such that $\lVert\delta\rVert = 1$ is called a *direction*	39
U^δ	the set $\{u_j^\delta = \langle u_i, \delta\rangle \mid u_i \in U\,\}$	40
f^δ	the transition map $f^\delta : M^{k_\delta} \to M$	41
D^δ	regular operator $D^\delta : M^{\mathbb{R}} \to M^{\mathbb{R}}$	41
H_δ^D	the set $\{v \in \mathbb{R}^2 \mid \langle\delta, v\rangle \geq R_{02}(D^\delta)\}$	42
σ_D	the set $\bigcap_\delta H_\delta^D$	42
$c_{R,R'}$	the *wedding cake*	53
$diam(\,)$	the *diameter* of a set	54

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CHAPTER I

INTRODUCTION

If you wish to observe bird life with some chance of obtaining interesting results, you should be somewhat familiar with birds, interested in birds, perhaps you should even like birds.

— George Polya

LET n and m be two positive integers. Consider, at first, the space $\mathbb{Z}^n$ and the finite set of states $M = \{0, 1, \ldots, m\}$. A map $x : \mathbb{Z}^n \to M$ is called a configuration and the set of all configurations is denoted by $M^{\mathbb{Z}^n}$. The image by the map x of an element $v \in \mathbb{Z}^n$ is denoted by x_v.

Let $V = \{v_1, \ldots, v_k\}$ be a finite list of elements of $\mathbb{Z}^n$. A map $f : M^k \to M$ is called a *transition map*. A set V and a transition map f with one and the same parameter k determine another map $P : M^{\mathbb{Z}^n} \to M^{\mathbb{Z}^n}$ by the following rule:

$$(Px)_p = f(x_{p+v_1}, \ldots, x_{p+v_k}) \quad \text{for all} \quad p \in \mathbb{Z}^n \,. \tag{1.1}$$

The t-th iteration of P, denoted by P^t, is defined by

$$P^0 = I \quad \text{and} \quad P^{t+1} = P \circ P^t \quad \text{for all} \quad t \in \mathbb{Z}_+ \,,$$

where I is the identity map.

The study of the dynamic system obtained by $\{P^t x\}_{t \in \mathbb{Z}_+}$, where x is a given configuration, is usually referred as *processes with local interaction*, *interacting particle systems*, *cellular automata* and by many other names. Searching internet for those terms and similar phrases, we get many thousands of results. In addition to that, the new research in this area is so abundant that surveys like Maes (2005) are in order.

We can trace cellular automata's way back to the 40s when John von Neumann designed a self-replicating machine. His construction was completed and published by Artur W. Birks in von Neumann (1966). The time was discrete, the space, in which this abstract machine functioned, also was discrete and two-dimensional and every component had 29 possible states and interacted only with its nearest neighbors.

Why are these topics so popular among modern scientists? Because they are very powerful as models of various kinds of reality. The phrase "programmable matter", first coined by Toffoli & Margolus (1991) is a good expression of this power. Indeed, processes with local interaction may be used to model an enormous range of natural phenomena.

Models of this sort typically include: a space, discrete or continuous; time, discrete or continuous; and the set of states of each component, also discrete or continuous. For example, many well-known works, including T. Liggett's fundamental book Liggett (2005), deal with models, where the space is discrete and one-dimensional, time is continuous and every component has two possible states.

A configuration x for which the set $\{ v \in \mathbb{Z}^n \, | x_v \neq 0 \}$ is finite is called an island. Any operator P is said to be an *eroder* if for every island x there is $\tau(x) \in \mathbb{Z}_+$ such that

$$(P^{\tau(x)}x)_v = 0 \quad \text{for all} \quad v \in \mathbb{Z}^n \, .$$

One may wonder whether an operator is an eroder. This issue is part of mathematicians' response to the following situation. Imagine a large uniform surface with a small defect. This may be a healthy biological tissue with just one sick cell or a large solid body, generally robust, but with a small defect. Which way will this situation take: may be this defect will disappear: the tissue may cure itself; or may be this defect will grow as an epidemy and contaminate all the tissue; or may be the defect will remain as it is, neither growing nor disappearing.

The power of such models has its opposite side: many questions about them have no algoritmic answer as presented in Kurdyumov (1980) and Petri (1987). So to obtain solvable problems we need somehow to restrict our range of systems. One useful restriction is monotonicity. We say that a transition map f is *monotonic* if

$$a_1 \leq b_1, \ldots, a_k \leq b_k \Longrightarrow f(a_1, \ldots, a_k) \leq f(b_1, \ldots, b_k).$$

An operator P is called *regular* if it is defined by (1.1) where f is monotonic and

$$f(a, \ldots, a) = a \quad \text{for all} \quad a \in M.$$

Some researches have investigated those regular operators, namely, Andrei Toom and Gregory Galperin.

1.1 Toom's approach

Andrei Toom has studied a regular operator P where $M = \{0,1\}$ in Toom (1976, 1980). There, he called a subset S of V by zero-set if

$$\forall\, w \in S \,:\, x_w = 0 \Longrightarrow f(x_{v_1}, \ldots, x_{v_k}) = 0\,.$$

Notice that there is just a finite number $l \in \mathbb{Z}_+$ of zero-sets: $Z_1, Z_2, \ldots, Z_l$. Each Z_j can be seen also as a subset of $\mathbb{R}^n$, whence, in that sense, we can consider its convex hull

$$conv(Z_j)\,.$$

Let us denote

$$\sigma_0 = \bigcap_{j=1,\ldots,l} conv(Z_j)\,.$$

Theorem [**Toom, 1980**]**:** *Regular operator P is an eroder if, and only if, $\sigma_0 = \emptyset$.*

1.2 Galperin's approach

Galperin (1975) considers regular operator where $n = 1$. There, Galperin defined a configuration x as increasing when

$$\forall v, w \in \mathbb{Z} : v < w \Longrightarrow x_v \leq x_w\,.$$

Decreasing configuration is defined similarly. If a configuration x is increasing or decreasing, there are $v_1, v_2 \in \mathbb{Z}$ and $a, b \in M$ such that $x_v = a$ for all $v < v_1$ and $x_v = b$ for all $v > v_2$. Whenever $a \neq b$, we say that x is a (a,b)-ladder.

Given a (a,b)-ladder x,

$$\mathcal{L}'(x) = \max\{\, v \in \mathbb{Z} \,|\, x_v = a\,\} \quad \text{and} \quad \mathcal{R}'(x) = \min\{\, v \in \mathbb{Z} \,|\, x_v = b\,\}\,.$$

We call by (a,b)-jump and denote by $j_{a,b}$ any (a,b)-ladder for which $\mathcal{R}'(x) - \mathcal{L}'(x) = 1$.

Galperin proved existence of the following limits:

$$\lim_{t\to\infty} \frac{\mathcal{L}'(P^t j_{a,b})}{t} = L_{a,b} \quad \text{and} \quad \lim_{t\to\infty} \frac{\mathcal{R}'(P^t j_{a,b})}{t} = R_{a,b}\,. \tag{1.2}$$

He called them left (a, b)-rate and right (a, b)-rate of regular operator P respectively and presented the following result:

Theorem [Galperin, 1975]: *A regular operator P where $n = 1$ and $M = \{0, 1, \ldots, m\}$ is an eroder if and only if*

$$R_{0,m} > L_{m,0}\,,\; R_{0,m-1} > L_{m-1,0}\,,\; \ldots\,,\; R_{0,1} > L_{1,0}\,.$$

1.3 In continuous space

Two cases with discrete space have been discussed: when $M = \{0, 1\}$ and when $n = 1$. Going beyond these results, the first case we meet is when $M = \{0, 1, 2\}$ and $n = 2$. Until now the only contribution to the research of this case was a study of one concrete example by Lima de Menezes & Toom (2006) and even it was useful: it showed behaviors which are impossible in the previously studied cases.

Unlike Toom (1976) and Galperin (1975, 1977) results, this work deals with continuous space. We consider the *space* $\mathbb{R}^n$ and its elements are called *points*. Elements of the set $M = \{0, 1, 2\}$ are called *states* and we order M by the evident rule $0 < 1 < 2$. Maps $x : \mathbb{R}^n \to M$ are called $\mathbb{R}^n$*-configurations* and the set of all $\mathbb{R}^n$-configurations is denoted by $M^{\mathbb{R}^n}$. Given a point $p \in \mathbb{R}^n$ its image by the $\mathbb{R}^n$-configuration x is denoted either by $x(p)$ or by x_p.

Any map $D : M^{\mathbb{R}^n} \to M^{\mathbb{R}^n}$ is called an *operator*. In particular, for every $q \in \mathbb{R}^n$ one can define an operator $S^q : M^{\mathbb{R}^n} \to M^{\mathbb{R}^n}$ given by

$$(S^q x)_p = x_{p-q} \quad \text{for all} \quad p \in \mathbb{R}^n\,.$$

An operator S^q is called a *shift* by q. An operator D is called *shift-invariant* if it commutes with all shifts of $M^{\mathbb{R}^n}$, i.e.,

$$D \circ S^q = S^q \circ D \quad \text{for all} \quad q \in \mathbb{R}\,,$$

where $\circ$ denotes the composition of maps.

We choose a *neighborhood*, that is a finite non-empty list

$$U = \{u_1, \ldots, u_k\}$$

of elements of $\mathbb{R}^n$. The non-negative number

$$r_U = \max\{\, |u_i| \,|\, u_i \in U\} \tag{1.3}$$

is called *neighborhood radius*.

Any map $f : M^k \to M$ is called a *transition map*. A neighborhood U and a transition map f with one and the same parameter k determine an *operator* D by the following rule:

$$(Dx)_p = f(x_{p+u_1}, \dots, x_{p+u_k}) \quad \text{for all} \quad p \in \mathbb{R}^n \,. \tag{1.4}$$

An operator D defined by (1.4) is shift-invariant. Indeed,

$$(S^q \circ Dx)_p = (Dx)_{p-q} = f(x_{p-q+u_1}, \dots, x_{p-q+u_k})$$

$$= f((S^q x)_{p+u_1}, \dots, (S^q x)_{p+u_k}) = (D \circ S^q x)_p.$$

Given two $\mathbb{R}^n$-configurations x and x', we write $x \prec x'$ and say that x *preceeds* x' if

$$x_p \leq x'_p \quad \text{for all} \quad p \in \mathbb{R}^n.$$

We call an operator D *monotonic* if

$$x \prec x' \Longrightarrow Dx \prec Dx'.$$

Notice that operator D defined by (1.4) is monotonic if and only if its transition map f is monotonic.

We call an operator D *regular* if it is defined by (1.4) where f is monotonic and

$$f(a, \dots, a) = a \qquad \text{for all} \qquad a \in M. \tag{1.5}$$

If D and D' are regular, then $\tilde{D} = D \circ D'$ is also regular.

For instance, each shift operator S^q is a regular operator. Moreover, if the neighborhood U has just one element then a regular operator is necessarily a shift operator. However, if the neighborhood U contains more than one element, then one can define a regular operator that is not shift operator. Consider a monotonic transition function $f : M^2 \to M$ where $f(0,2) = f(2,0) = 2$ and $U = \{0,1\}$. The configuration

$$x(v) = \begin{cases} 2 & \text{if} \quad v = 0, \\ 0 & \text{otherwise}, \end{cases} \tag{1.6}$$

has $Dx \neq S^w x$ for any $w \in \mathbb{R}$.

Nevertheless, we will see later that under some appropriate assumption each regular operator D acts like a shift operator on a special class of configurations. All our main results concern regular operators.

For any $\mathbb{R}^n$-configuration x we denote

$$max(x) = \max\{\, x_p \,|\, p \in \mathbb{R}^n \,\}\,.$$

Similarly the t-th iterated of D, D^t, is defined by

$$D^0 = S^0 \qquad \text{and} \qquad D^{t+1} = D \circ D^t \qquad \text{for all} \qquad t \in \mathbb{Z}_+\,.$$

We say that a regular operator D 2-*degrades* a $\mathbb{R}^n$-configuration x if there is $t_0 \in \mathbb{Z}_+$ such that $max(D^{t_0}x) < 2$ (whence, since D is regular, $max(D^t x) < 2$ for every $t \geq t_0$). Accordingly, we say that D does not 2-degrade a $\mathbb{R}^n$-configuration x if $max(D^t x) = 2$ for all $t \in \mathbb{Z}_+$.

If D 2-degrades a $\mathbb{R}^n$-configuration x, then we define the 2-*lifetime* of x under D as the non-negative number

$$\tau_2^D(x) = \min\{\, t \in \mathbb{Z}_+ \,|\, max(D^t x) < 2\}.$$

Otherwise, we say that the 2-lifetime of x is infinity and write $\tau_2^D(x) = \infty$.

We call $x \in M^{\mathbb{R}^n}$ an *island* if the set $\{p \in \mathbb{R}^n \,|\, x_p \neq 0\}$ is bounded. Notice that if D is regular and x is an island, then $D^t x$ is also an island. An operator is called a 2-*degrader* if it 2-degrades all the islands. Accordingly, an operator is a *non*-2-*degrader* if there is an island which it does not 2-degrade.

We shall concentrate our attention on the island $d_R : \mathbb{R}^n \to M$ depending on a non-negative parameter R and defined by

$$d_R(p) = \begin{cases} 2 & \text{if} \quad \|p\| \leq R\,, \\ 0 & \text{otherwise,} \end{cases} \tag{1.7}$$

where $\| \, \|$denotes the usual norm[1].

An operator D is said to be *linear 2-degrader* if there exist positive real numbers λ_D and α_D such that

$$\tau_2^D(d_R) \leq \lambda_D R + \alpha_D\,.$$

Chapter 2 deals with the one-dimensional case, i.e., the continuous space $\mathbb{R}$. There, we have generalized Galperin's $L_{0,2}$, $R_{0,2}$, $L_{2,0}$ and $R_{2,0}$ given in (1.2). There are some technical difficulties, but the main results are similar to those of Galperin. In particular, we shall see that an one-dimensional 2-degrader is always a linear 2-degrader. Chapter 3 presents a sufficient condition for a two-dimensional regular operator to be a linear 2-degrader. It is proved using

[1] The usual norm $\| \, \| : \mathbb{R}^n \to \mathbb{R}_+$ is given by $\|p\| = \sqrt{p_1^2 + \cdots + p_n^2}$.

one-dimensional tools introduced in Chapter 2. Chapter 4 presents a sufficient condition for a 2-degrader not to be a linear 2-degrader. Moreover, it also shows that under some assumptions over a regular operator there is a island that grows without a bound.

There is no doubt to me that the beauty emerging from the studying of 2-degrader character sufficiently justifies this work.

CHAPTER 2

VELOCITIES À LA GALPERIN IN $\mathbb{R}$

The teacher should encourage the students to imagine cases in which they could utilize again the procedure used, or apply the result obtained. Can you use the result, or the method, for some other problem?

— George Polya

PROFESSOR Andrei Toom once told the author (in portuguese):

"Isto pode ser feito em $\mathbb{R}$ [referindo-se a Galperin (1975)]... E isto é bom fazer."[1]

This Chapter investigates how to apply Galperin's method to continuous space $\mathbb{R}$ and the set of states $M = \{0, 1, 2\}$.

À la Galperin, a configuration $x \in M^{\mathbb{R}}$ is called *increasing* if

$$\forall\, p, q \in \mathbb{R} \,:\, p < q \Longrightarrow x_p \leq x_q.$$

Similarly, a configuration $x \in M^{\mathbb{R}}$ is said to be *decreasing* if

$$\forall\, p, q \in \mathbb{R} \,:\, p > q \Longrightarrow x_p \leq x_q.$$

We say that a configuration is *monotone* if it is increasing or decreasing. Notice that if x is monotone, then there are $p_1, p_2 \in \mathbb{R}$ and $a, b \in M$ such that

$$x_p = a \quad \text{for all} \quad p < p_1 \quad \text{and} \quad x_p = b \quad \text{for all} \quad p > p_2\,.$$

[1] "It can be done in $\mathbb{R}$... And it should be done."

A monotone configuration x for which $a \neq b$ is said to be an (ab)-*ladder*. Figure 2.1 presents a (02)-ladder x and a (21)-ladder x'.

For any (ab)-ladder x, we denote

$$\mathcal{L}(x) = \sup\{\, p \in \mathbb{R} \,|\, x_p = a \,\} \quad \text{and} \quad \mathcal{R}(x) = \inf\{\, p \in \mathbb{R} \,|\, x_p = b \,\}\,.$$

The real numbers $\mathcal{L}(x)$ and $\mathcal{R}(x)$ are said to be the *left coordinate* and *right coordinate* respectively. Furthermore, the non-negative number

$$l(x) = \mathcal{R}(x) - \mathcal{L}(x)$$

is called the *length* of the (ab)-ladder x.

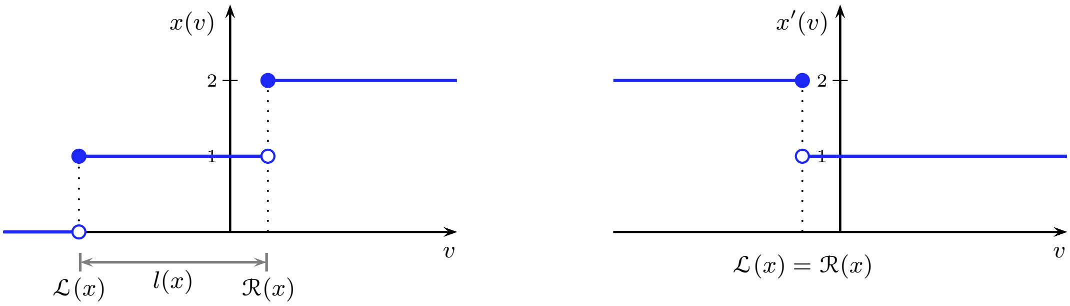

Figure 2.1: *Example of a* (02)*-ladder* x *and a* (21)*-ladder* x'.

We say that an (ab)-ladder x is *right-continuous at* $p_0 \in \mathbb{R}$ if there is a positive real number ϵ such that

$$p_0 < p < p_0 + \epsilon \Longrightarrow x(p) = x(p_0)\,.$$

Moreover, an (ab)-ladder x is said to be *right-continuous* if it is right-continuous at all $p \in \mathbb{R}$. A *left-continuous* ladder is similarly defined. Notice that configurations x and x' in Figure 2.1 are right-continuous and left-continuous respectively.

An (ab)-ladder x with $l(x) = \mathcal{L}(x) = \mathcal{R}(x) = 0$ that is either increasing and right-continuous or decreasing and left-continuous is called an (ab)-*jump* and it is denoted by j_{ab}. Figure 2.2 shows j_{02} and j_{10}.

From Lemma 1, which is proved in the Section 2.1, there are $V_{01}, V_{12} \in \mathbb{R}$ for which

$$Dj_{01} = S^{V_{01}} j_{01} \qquad \text{and} \qquad Dj_{12} = S^{V_{12}} j_{12}.$$

The real numbers V_{01} and V_{12} are called the (01)-*velocity* and the (12)-*velocity* of the one-dimensional regular operator D respectively.

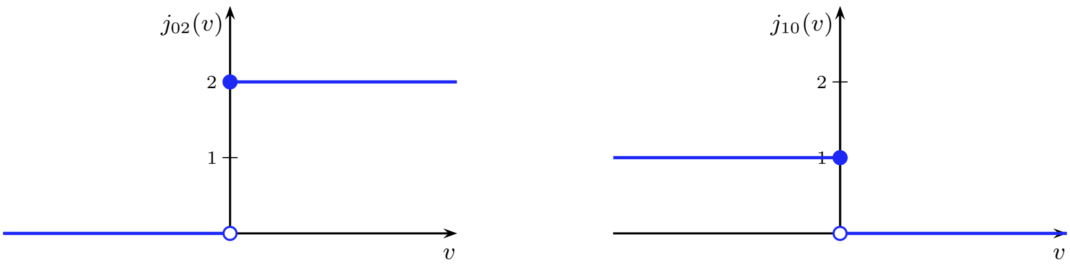

Figure 2.2: (02)-*jump and* (10)-*jump.*

The main results presented in this text are called Theorems. There are four of them in Chapter 2, namely,

Theorem 1 *Let D be a one-dimensional regular operator. Then the following limits exist:*

$$\lim_{t\to\infty} \frac{\mathcal{L}(D^t j_{02})}{t} = L_{02}(D)\,, \quad \lim_{t\to\infty} \frac{\mathcal{R}(D^t j_{02})}{t} = R_{02}(D)\,,$$

$$\lim_{t\to\infty} \frac{\mathcal{L}(D^t j_{20})}{t} = L_{20}(D) \quad \textit{and} \quad \lim_{t\to\infty} \frac{\mathcal{R}(D^t j_{20})}{t} = R_{20}(D)\,.$$

The limits $L_{02}(D)$, $R_{02}(D)$, $L_{20}(D)$ and $R_{20}(D)$ are called *left* (02)-*velocity*, *right* (02)-*velocity*, *left* (20)-*velocity* and *right* (20)-*velocity* of the regular operator D respectively. They are natural generalizations of the limits presented in (1.2). We venture to simplify the notation of these limits to L_{02}, R_{02}, L_{20} and R_{20} whenever it does not cause confusion.

Theorem 2 *Let D be a one-dimensional regular operator. Then there are a right-continuous* (02)-*ladder x and a left-continuous* (20)-*ladder x' for which*

$$\mathcal{R}(D^t x) = \mathcal{R}(x) + tR_{02} \quad \textit{and} \quad \mathcal{L}(D^t x') = \mathcal{L}(x') + tL_{20} \quad \textit{for all} \quad t \in \mathbb{Z}_+\,.$$

One may say that there is always a right-continuous (02)-ladder x for which the sequence of right coordinates $\{\mathcal{R}(D^t x)\}_{t\in\mathbb{Z}_+}$ has a *uniform motion*.

Theorem 3 *If $V_{01} > V_{12}$, then*

$$-2r_U + tV_{02} \le \mathcal{L}(D^t j_{02}) \le \mathcal{R}(D^t j_{02}) \le 2r_U + tV_{02} \quad \textit{for all} \quad t \in \mathbb{Z}_+\,,$$

and $V_{02} = L_{02} = R_{02}$.

Theorem 4 *For any one-dimensional regular operator D exactly one of these two cases takes place:*

- ▷ *Operator D is a linear 2-degrader.*
- ▷ *Operator D is a non-2-degrader.*

In the one-dimensional case, a 2-degrader is always a linear 2-degrader.

Theorems 1, 2, 3 and 4 are proved in Sections 2.1, 2.2, 2.3 and 2.4 respectively. Indeed the main ideas to prove these results come from Galperin. However, instead of saying that Galperin's ideas could be adapted, for the reader's convenience we wrote complete proofs down.

2.1 Proof of Theorem 1 [existence of velocities]

Lemma 1 *Let D be a one-dimensional regular operator and x be a right-continuous increasing (ab)-ladder. Then $D^t x$ is also a right-continuous increasing (ab)-ladder.*

Proof: Let p and p' be two points such that $p < p'$. Since x is increasing, then

$$x_{p+u_1} \leq x_{p'+u_1} \,,\ \ldots\ ,\ x_{p+u_k} \leq x_{p'+u_k} \,.$$

Hence, from monotonicity of f,

$$(Dx)_p = f(x_{p+u_1}, \ldots, x_{p+u_k}) \leq f(x_{p'+u_1}, \ldots, x_{p'+u_k}) = (Dx)_{p'} \,.$$

Thus Dx is also increasing.

Notice that

$$x_p = a \quad \text{for all} \quad p < \mathcal{L}(x) \quad \text{and} \quad x_p = b \quad \text{for all} \quad p \geq \mathcal{R}(x) \,.$$

Therefore, from (1.5),

$$(Dx)_p = a \quad \text{for all} \quad p < \mathcal{L}(x) - r_U \quad \text{and} \quad (Dx)_p = b \quad \text{for all} \quad p \geq \mathcal{R}(x) + r_U \,.$$

Hence Dx is an (ab)-ladder.

Now, let us prove that Dx is right-continuous. Let p be a point in $\mathbb{R}$ and j be an element of $\{1, 2, \ldots, k\}$. Since x is right-continuous, there is a positive real number ϵ_j such that

$$p_0 + u_j < p + u_j < p_0 + u_j + \epsilon_j \Longrightarrow x(p_0 + u_j) = x(p + u_j) \,.$$

Define $\epsilon = \min\{\, \epsilon_j \,|\, j \in \{1, 2, \ldots, k\}\,\}$. Therefore

$$p_0 < p < p_0 + \epsilon \Longrightarrow x(p_0 + u_j) = x(p + u_j) \quad \text{for all} \quad j \in \{1, 2, \ldots, k\} \,. \qquad (2.1)$$

Thus, from (2.1)

$$p_0 < p < p_0 + \epsilon \Longrightarrow (Dx)_{p_0} = f(x_{p_0+u_1}, \ldots, x_{p_0+u_k}) = f(x_{p+u_1}, \ldots, x_{p+u_k}) = (Dx)_p \,.$$

Since p_0 is arbitrary, Dx is right-continuous. So, it is true for $t = 1$.
Suppose that $D^t x$ is a right-continuous increasing (ab)-ladder for an arbitrary natural $t > 1$. Hence, from the case for $t = 1$, $D^{t+1}x = D(D^t x)$ is also a right-continuous increasing (ab)-ladder. Thus it is true for $t + 1$.

Lemma 1 is proved.

From Lemma 1, one might say that the set of right-continuous (ab)-ladders is *invariant*, or *stable*, under the one-dimensional regular operator D. We may call Lemma 1 the *Invariance's Lemma*.

Lemma 2 *Let x be an (ab)-ladder. Then $\mathcal{L}(x) \leq \mathcal{R}(x)$.*

Proof: Suppose that x is increasing. At first, let us prove the following assertion:
Let p' be an element of $\{\, p \in \mathbb{R} \,|\, x_p = b \,\}$ and $\tilde{p}$ be an element of $\{\, p \in \mathbb{R} \,|\, x_p = a \,\}$. Then $\tilde{p} < p'$.
Indeed, suppose that $\tilde{p} > p'$. Since x is increasing, then $a = x_{\tilde{p}} \geq x_{p'} = b$. It is a contradiction. Thus the assertion is proved.
Therefore

$$\sup\{\, p \in \mathbb{R} \,|\, x_p = a \,\} \leq p' \quad \text{for all} \quad p' \in \{\, p \in \mathbb{R} \,|\, x_p = b \,\} \,.$$

Hence

$$\sup\{\, p \in \mathbb{R} \,|\, x_p = a \,\} \leq \inf\{\, p \in \mathbb{R} \,|\, x_p = b \,\} \,.$$

The proof for a decreasing ladder is similar.

Lemma 2 is proved.

Lemma 3 *Let x and x' be increasing (ab)-ladders such that $x \prec x'$. Then*

$$\mathcal{L}(x') \leq \mathcal{L}(x) \quad \textit{and} \quad \mathcal{R}(x') \leq \mathcal{R}(x) \,.$$

Proof: Let $\hat{p}$ be an arbitrary element of $\{\, p \in \mathbb{R} \,|\, x'_p = a \,\}$. If $x \prec x'$, then $x'_{\hat{p}} \geq x_{\hat{p}}$. Hence $x_{\hat{p}} = a$.
So, we have just proved that

$$\{\, p \in \mathbb{R} \,|\, x'_p = a \,\} \subset \{\, p \in \mathbb{R} \,|\, x_p = a \,\} \,,$$

whence

$$\sup\{\, p \in \mathbb{R} \,|\, x'_p = a \,\} \le \sup\{\, p \in \mathbb{R} \,|\, x_p = a \,\} .$$

The proof for $\mathcal{R}(x') \le \mathcal{R}(x)$ is similar.

Lemma 3 is proved.

Lemma 4 *Let x be an (ab)-ladder. Then*

$$\mathcal{L}((S^q)^t x) = \mathcal{L}(x) + tq \quad \textit{and} \quad \mathcal{R}((S^q)^t x) = \mathcal{R}(x) + tq \quad \textit{for all} \quad t \in \mathbb{Z}_+ .$$

Proof: By definition

$$\mathcal{L}(S^q x) = \sup\{\, p \in \mathbb{R} \,|\, (S^q x)_p = a \,\} = \sup\{\, p \in \mathbb{R} \,|\, x_{p-q} = a \,\}$$

Let us define $p' = p - q$. Then

$$\mathcal{L}(S^q x) = \sup\{\, p' + q \in \mathbb{R} \,|\, x_{p'} = a \,\} = q + \sup\{\, p' \in \mathbb{R} \,|\, x_{p'} = a \,\} = q + \mathcal{L}(x).$$

So, it is true for $t = 1$.

Suppose that it is true for an arbitrary natural $t > 1$. Hence, from the case where $t = 1$,

$$\mathcal{L}(S^q((S^q)^t x)) = q + \mathcal{L}((S^q)^t x).$$

At last, from the inductive hypothesis

$$\mathcal{L}((S^q)^{t+1} x) = q + \mathcal{L}(x) + tq = \mathcal{L}(x) + (t+1)q.$$

Thus it is true for $t + 1$.

Lemma 4 is proved.

Notice that Lemma 4 is a particular case of Theorem 2.

Galperin (1975) contains the following lemma:

Lemma *If $\{A_t\}_{t \in \mathbb{Z}_+}$ is a sequence of real numbers satisfying one of these two conditions:*

- ▷ $A_{t+\tau} \le A_t + A_\tau$ *for all* $t, \tau \in \mathbb{Z}_+$.
- ▷ $A_{t+\tau} \ge A_t + A_\tau$ *for all* $t, \tau \in \mathbb{Z}_+$.

Then the limit $\lim_{t\to\infty} A_t / t$ exists.

Proof of Theorem 1: Let t be an arbitrary element of $\mathbb{Z}_+$. From Lemma 1, $D^t j_{02}$ is a right-continuous (02)-ladder. Therefore

$$S^{\mathcal{R}(D^t j_{02})} j_{02} \prec D^t j_{02} \prec S^{\mathcal{L}(D^t j_{02})} j_{02} . \tag{2.2}$$

Let τ be an element of $\mathbb{Z}_+$. Since D is shift-invariant and monotonic,

$$S^{\mathcal{R}(D^t j_{02})} \circ D^\tau j_{02} \prec D^{t+\tau} j_{02} \prec S^{\mathcal{L}(D^t j_{02})} \circ D^\tau j_{02}\,. \tag{2.3}$$

From (2.2),

$$S^{\mathcal{R}(D^\tau j_{02})} j_{02} \prec D^\tau j_{02} \prec S^{\mathcal{L}(D^\tau j_{02})} j_{02}\,. \tag{2.4}$$

From (2.4) and monotonicity

$$S^{\mathcal{R}(D^t j_{02})} \circ S^{\mathcal{R}(D^\tau j_{02})} j_{02} \prec S^{\mathcal{R}(D^t j_{02})} \circ D^\tau j_{02} \tag{2.5}$$

and

$$S^{\mathcal{L}(D^t j_{02})} \circ D^\tau j_{02} \prec S^{\mathcal{L}(D^t j_{02})} \circ S^{\mathcal{L}(D^\tau j_{02})} j_{02}\,, \tag{2.6}$$

From (2.3), (2.5) and (2.6)

$$S^{\mathcal{R}(D^t j_{02})+\mathcal{R}(D^\tau j_{02})} j_{02} \prec D^{t+\tau} j_{02} \prec S^{\mathcal{L}(D^t) j_{02}+\mathcal{L}(D^\tau j_{02})} j_{02}\,.$$

From Lemma 3,

$$\mathcal{R}(S^{\mathcal{R}(D^t j_{02})+\mathcal{R}(D^\tau j_{02})} j_{02}) \geq \mathcal{R}(D^{t+\tau} j_{02}) \tag{2.7}$$

and

$$\mathcal{L}(D^{t+\tau} j_{02}) \geq \mathcal{L}(S^{\mathcal{L}(D^t j_{02})+\mathcal{L}(D^\tau j_{02})} j_{0,2})\,. \tag{2.8}$$

From Lemma 4 and (2.7)

$$\mathcal{R}(D^{t+\tau} j_{02}) \leq \mathcal{R}(D^t j_{02}) + \mathcal{R}(D^\tau j_{02}) \quad \text{for all} \quad t, \tau \in \mathbb{Z}_+$$

and from Lemma 4 and (2.8)

$$\mathcal{L}(D^{t+\tau} j_{02}) \geq \mathcal{L}(D^t j_{02}) + \mathcal{L}(D^\tau j_{02}) \qquad \text{for all} \qquad t, \tau \in \mathbb{Z}_+\,.$$

The existence of L_{02} and R_{02} follows from the previous Lemma.

The proof for L_{20} and R_{20} is similar.

Theorem 1 is proved.

Notice, from Lemma 2, that

$$L_{02} \leq R_{02} \quad \text{and} \quad L_{20} \leq R_{20}\,.$$

In some special cases we may have $L_{02} = R_{02}$ or $L_{20} = R_{20}$. For instance, a shift operator S^q has $L_{02} = R_{02} = q$. Indeed, from Lemma 4

$$\frac{\mathcal{L}((S^q)^t x)}{t} = \frac{\mathcal{L}(x) + tq}{t} \quad \text{and} \quad \frac{\mathcal{R}((S^q)^t x)}{t} = \frac{\mathcal{R}(x) + tq}{t} \quad \text{for all} \quad t \in \mathbb{Z}_+\,.$$

Thus

$$\lim_{t\to\infty} \frac{\mathcal{L}((S^q)^t j_{02})}{t} = q = \lim_{t\to\infty} \frac{\mathcal{R}((S^q)^t j_{02})}{t} \,.$$

Moreover, from Lemma 13, $L_{02} = R_{02}$ whenever $V_{01} > V_{12}$.

Lemma 5 *Let D be a one-dimensional regular operator. Define $\tilde{D} = S^q \circ D$. Then*

$$L_{02}(\tilde{D}) = q + L_{02}(D) \quad \textit{and} \quad R_{02}(\tilde{D}) = q + R_{02}(D)\,.$$

Proof: From shift-invariance

$$\tilde{D}^t = (S^q)^t \circ D^t$$

and from Lemma 4

$$\mathcal{L}(\tilde{D}^t j_{02}) = \mathcal{L}((S^q)^t \circ D^t j_{02}) = \mathcal{L}(D^t j_{02}) + tq\,.$$

Thus

$$\lim_{t\to\infty} \frac{\mathcal{L}(\tilde{D}^t j_{02})}{t} = q + L_{02}(D)\,.$$

The proof that $R_{02}(\tilde{D}) = q + R_{02}(D)$ is similar.

Lemma 5 is proved.

Lemma 5 is not employed in that Chapter 2, but it will be useful in Chapter 3.

2.2 Proof of Theorem 2

Lemma 6 *Let x be a right-continuous (02)-ladder with $l(x) > 2r_U$. Then*

$$\mathcal{L}(Dx) = \mathcal{L}(x) + V_{01} \quad \textit{and} \quad \mathcal{R}(Dx) = \mathcal{R}(x) + V_{12}\,.$$

Moreover, $l(Dx) = l(x) + (V_{12} - V_{01})$.

Proof: Notice that

$$x_p = (S^{\mathcal{L}(x)} j_{01})_p \quad \text{for all} \quad p < \mathcal{L}(x) + 2r_U$$

and

$$x_p = (S^{\mathcal{R}(x)} j_{12})_p \quad \text{for all} \quad p \geq \mathcal{R}(x) - 2r_U\,.$$

See Figure 2.3.

Hence, since D is defined by (1.4),

$$(Dx)_p = (DS^{\mathcal{L}(x)} j_{01})_p \quad \text{for all} \quad p < \mathcal{L}(x) + r_U$$

and

$$(Dx)_p = (DS^{\mathcal{R}(x)} j_{12})_p \quad \text{for all} \quad p \geq \mathcal{R}(x) - r_U \,.$$

Moreover

$$(Dx)_p = 1 \quad \text{for all} \quad \mathcal{L}(x) + r_U \leq p < \mathcal{R}(x) - r_U \,.$$

Therefore,

$$\mathcal{L}(Dx) = \sup\{\, p \in \mathbb{R} \,|\, (Dx)_p = 0 \,\} = \sup\{\, p \in (-\infty, \mathcal{L}(x) + r_U) \,|\, (Dx)_p = 0 \,\}$$

$$= \sup\{\, p \in (-\infty, \mathcal{L}(x) + r_U) \,|\, (DS^{\mathcal{L}(x)} j_{01})_p = 0 \,\}$$

$$= \sup\{\, p \in \mathbb{R} \,|\, (DS^{\mathcal{L}(x)} j_{01})_p = 0 \,\} = \mathcal{L}(S^{\mathcal{L}(x)} Dj_{01}) \,.$$

From Lemma 4

$$\mathcal{L}(S^{\mathcal{L}(x)} Dj_{01}) = \mathcal{L}(x) + \mathcal{L}(Dj_{01}) = \mathcal{L}(x) + \mathcal{L}(S^{V_{01}} j_{01}) = \mathcal{L}(x) + V_{01} \,.$$

Similarly, we can prove that $\mathcal{R}(Dx) = \mathcal{R}(x) + V_{12}$.

Thus

$$l(Dx) = \mathcal{R}(Dx) - \mathcal{L}(Dx) = l(x) + (V_{12} - V_{01}) \,.$$

Lemma 6 is proved.

Lemma 7 *Let x be a right-continuous (02)-ladder with $l(x) > 2r_U$. If $V_{01} \leq V_{12}$, then*

$$\mathcal{L}(D^t x) = \mathcal{L}(x) + tV_{01} \quad \textit{and} \quad \mathcal{R}(D^t x) = \mathcal{R}(x) + tV_{12} \quad \textit{for all} \quad t \in \mathbb{Z}_+ \,.$$

Proof: From Lemma 6, it is true for $t = 1$.

Suppose that it is true for an arbitrary natural $t > 1$, i.e.,

$$\mathcal{L}(D^t x) = \mathcal{L}(x) + tV_{01} \quad \text{and} \quad \mathcal{R}(D^t x) = \mathcal{R}(x) + tV_{12} \,. \tag{2.9}$$

Hence

$$l(D^t x) = \mathcal{R}(x) - \mathcal{L}(x) + t(V_{12} - V_{01}) > 2r_U \,.$$

From Lemma 1, $D^t x$ is a right-continuous (02)-ladder. Then, from Lemma 6,

$$\mathcal{L}(D(D^t x)) = \mathcal{L}(D^t x) + V_{01} \,. \tag{2.10}$$

Hence, from (2.9) and (2.10),

$$\mathcal{L}(D^{t+1} x) = \mathcal{L}(x) + (t+1)V_{01} \,.$$

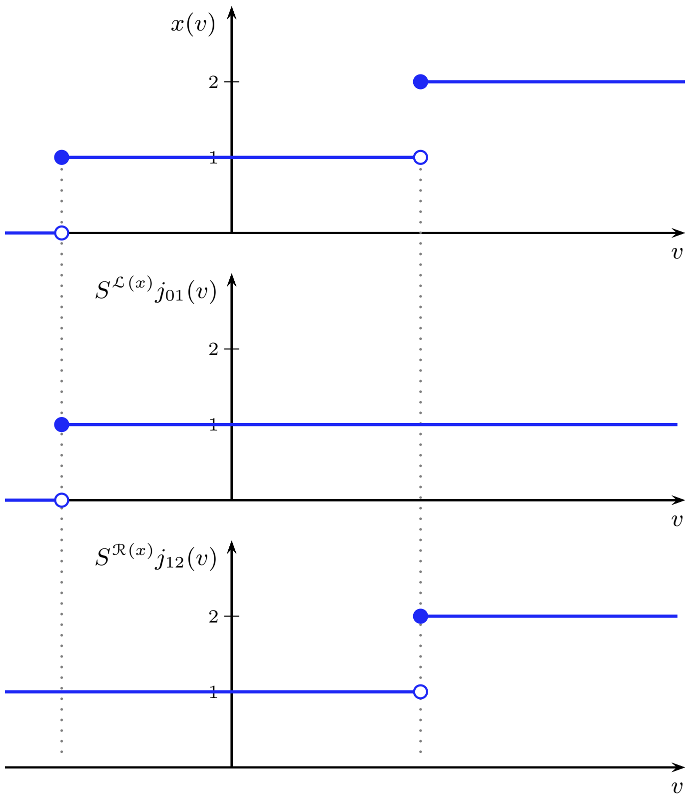

Figure 2.3: *Configurations x, $S^{\mathcal{L}(x)}j_{01}$ and $S^{\mathcal{R}(x)}j_{12}$.*

Similarly, one can prove that $\mathcal{R}(D^{t+1}x) = \mathcal{R}(x) + (t+1)V_{12}$. So it is true for $t+1$.

Lemma 7 is proved.

Let us remember a classical result of real analysis known as *Sandwich Theorem* or

Squeeze Theorem *Let $\{a_t\}_{t\in\mathbb{Z}_+}$, $\{b_t\}_{t\in\mathbb{Z}_+}$ and $\{c_t\}_{t\in\mathbb{Z}_+}$ be sequences of real numbers. Suppose that*

$$\lim_{t\to\infty} a_t = \lim_{t\to\infty} b_t = l$$

and $a_t \leq c_t \leq b_t$ for all $t \in \mathbb{Z}_+$. Then

$$\lim_{t\to\infty} c_t = l\,.$$

Lemma 8 *Let D be a one-dimensional regular operator for which $V_{01} \leq V_{12}$. Then*

$$L_{02} = V_{01} \quad \textit{and} \quad R_{02} = V_{12}\,.$$

Proof: Let x be a right-continuous (02)-ladder with $l(x) > 2r_U$. There are $q, q' \in \mathbb{R}$ for which

$$S^q x \prec j_{02} \prec S^{q'} x\,.$$

From monotonicity and shift-invariance

$$S^q D^t x \prec D^t j_{02} \prec S^{q'} D^t x \,.$$

From Lemma 3

$$\mathcal{R}(S^q D^t x) \geq \mathcal{R}(D^t j_{02}) \geq \mathcal{R}(S^{q'} D^t x) \,,$$

from Lemma 4

$$q + \mathcal{R}(D^t x) \geq \mathcal{R}(D^t j_{02}) \geq q' + \mathcal{R}(D^t x)$$

and from Lemma 6

$$\frac{q + \mathcal{R}(x) + tV_{12}}{t} \geq \frac{\mathcal{R}(D^t j_{02})}{t} \geq \frac{q' + \mathcal{R}(x) + tV_{12}}{t} \quad \text{for all} \quad t \in \mathbb{Z}_+ \,.$$

Thus, from the Squeeze Theorem, $R_{02} = V_{12}$.

Similarly, we can prove that $L_{02} = V_{01}$.

Lemma 8 is proved.

Notice that Lemma 8 gave us an alternative proof for the existence of L_{02} and R_{02} in the case where $V_{01} \leq V_{12}$.

Let us denote the set of all right-continuous (02)-ladders by X. Two right-continuous (02)-ladders x and x' are regarded as ***equivalent***, and written $x \sim x'$, if there is $q \in \mathbb{R}$ such that $x' = S^q x$. Notice that right-continuous (02)-ladders x and x' are equivalent if and only if $l(x) = l(x')$.

We denote by $[x]$ the equivalence class of the right-continuous (02)-ladder x, i.e.,

$$[x] = \{\, x' \in X \,|\, l(x') = l(x) \,\} \,.$$

The set of all those equivalence classes is denoted by Y, i.e.,

$$Y = \{\, [x] \,|\, x \in X \,\}.$$

Notice that the map $\rho : Y \times Y \to \mathbb{R}_+$ given by

$$\rho([x],[x']) = |l(x) - l(x')|$$

defines a metric. Indeed,

▷ $[x] = [x'] \Leftrightarrow l(x) = l(x') \Leftrightarrow |l(x) - l(x')| = \rho([x],[x']) = 0.$

▷ $\rho([x],[x']) = |l(x) - l(x')| - |l(x') - l(x)| = \rho([x'],[x]).$

▷ $\rho([x],[x'']) = |l(x) - |(x'')| \leq |l(x) - l(x')| + |l(x') - l(x'')| = \rho([x],[x']) + \rho([x'],[x'']).$

Lemma 9 *The map $\psi : Y \to \mathbb{R}_+$ given by $\psi([x]) = l(x)$ is continuous and bijective.*

Proof: The map ψ is continuous and bijective by construction.

Lemma 9 is proved.

From lemma 1 (Invariance's Lemma) $x \in X$ implies $Dx \in X$. Therefore, D induces the map $\widehat{D} : Y \to Y$ given by $\widehat{D}[x] = [Dx]$ as presented in the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{D} & X \\ {\scriptstyle [\cdot]}\big\downarrow & & \big\downarrow{\scriptstyle [\cdot]} \\ Y & \xrightarrow{\widehat{D}} & Y \end{array}$$

Lemma 10 *Let x and x' be elements of X. Define*

$$\overline{w} = \inf\{\, w \in \mathbb{R} \,|\, S^w x \prec x' \,\} \quad \textit{and} \quad \underline{w} = \sup\{\, w \in \mathbb{R} \,|\, x' \prec S^w x \,\}\,.$$

Then

$$\rho([x], [x']) = \overline{w} - \underline{w}\,.$$

Proof: At first, suppose that $l(x) \leq l(x')$. Notice that in such case $\underline{w}$ and $\overline{w}$ are such that

$$\mathcal{L}(S^{\underline{w}} x) = \mathcal{L}(x') \quad \text{and} \quad \mathcal{R}(S^{\overline{w}} x) = \mathcal{R}(x')\,.$$

as presented in Figure 2.4.

From Lemma 4

$$\underline{w} = \mathcal{L}(x') - \mathcal{L}(x) \quad \text{and} \quad \overline{w} = \mathcal{R}(x') - \mathcal{R}(x)\,.$$

Therefore

$$\overline{w} - \underline{w} = \mathcal{R}(x') - \mathcal{R}(x) - \mathcal{L}(x') + \mathcal{L}(x) = l(x') - l(x) = |l(x) - l(x')|\,.$$

Now, suppose that $l(x) > l(x')$. Then

$$\mathcal{R}(S^{\underline{w}} x) = \mathcal{R}(x') \quad \text{and} \quad \mathcal{L}(S^{\overline{w}} x) = \mathcal{L}(x')\,.$$

From Lemma 4

$$\underline{w} = \mathcal{R}(x') - \mathcal{R}(x) \quad \text{and} \quad \overline{w} = \mathcal{L}(x') - \mathcal{L}(x)\,.$$

Thus

$$\overline{w} - \underline{w} = l(x) - l(x') = |l(x) - l(x')|\,.$$

Lemma 10 is proved.

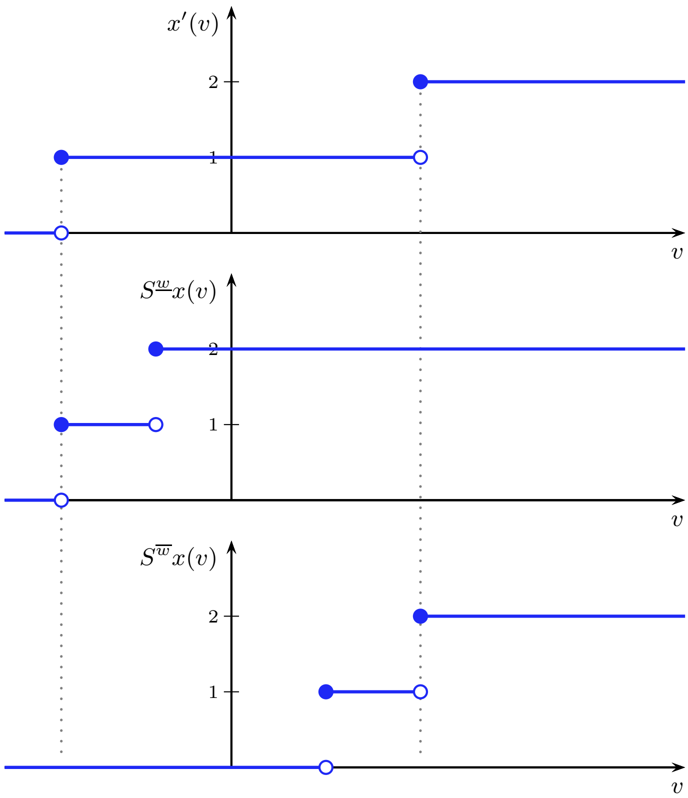

Figure 2.4: *Configurations x', $S^{\underline{w}}x$ and $S^{\overline{w}}x$.*

Lemma 11 *The map $\widehat{D} : Y \to Y$ satisfies*

$$\rho(\widehat{D}[x], \widehat{D}[x']) \leq \rho([x],[x']) \quad \textit{for all} \quad [x],[x'] \in Y\,.$$

In particular, $\widehat{D}$ is continuous.

Proof: Let x' be an element of $[x']$. There is $x \in [x]$ such that

$$S^{\rho([x],[x'])}x \prec x' \prec x$$

Since D is monotonic and shift-invariant,

$$S^{\rho([x],[x'])}Dx \prec Dx' \prec Dx$$

If $\overline{w} = \inf\{\, w \in \mathbb{R} \,|\, S^{w}Dx \prec Dx' \,\}$ and $\underline{w} = \sup\{\, w \in \mathbb{R} \,|\, Dx' \prec S^{w}Dx \,\}$, then

$$S^{\rho([x],[y])}Dx \prec S^{\overline{w}}Dx \prec Dx' \prec S^{\underline{w}}Dx \prec Dx\,.$$

From Lemma 3

$$\overline{w} \leq \rho([x],[x']) \quad \text{and} \quad \underline{w} \geq 0\,.$$

Thus, from Lemma 10,

$$\rho([Dx],[Dx']) = \overline{w} - \underline{w} \le \rho([x],[x'])\,.$$

Lemma 11 is proved.

Lemma 12 *Let x be a* (02)*-ladder with $l(x) \le 2r_U$. Then*

$$l(Dx) \le 4r_U\,.$$

Proof: Let x be a (02)-ladder with $l(x) \le 2r_U$. Since D is defined by (1.4),

$$(Dx)_p = 0 \quad \text{for all} \quad p < \mathcal{L}(x) - r_U$$

and

$$(Dx)_p = 2 \quad \text{for all} \quad p \ge \mathcal{R}(x) + r_U\,.$$

Therefore

$$\mathcal{L}(Dx) = \sup\{\, v \in \mathbb{R} \,|\, (Dx)_p = 0\,\} \ge \mathcal{L}(x) - r_U$$

and

$$\mathcal{R}(Dx) = \inf\{\, v \in \mathbb{R} \,|\, (Dx)_p = 2\,\} \le \mathcal{R}(x) + r_U\,.$$

Thus

$$\mathcal{R}(Dx) - \mathcal{L}(Dx) \le \mathcal{R}(x) + r_U - \mathcal{L}(x) + r_U = l(x) + 2r_U \le 4r_U\,.$$

Lemma 12 is proved.

Let us recall a simple and useful *Fixed Point Theorem*, namely,

Fixed point theorem *Let $\phi : [a,b] \to [a,b]$ be a continuous map. Then there exists $c \in [a,b]$ such that $\phi(c) = c$.*

Lemma 13 *Let D be a one-dimensional regular operator. If $V_{01} > V_{12}$, then there are $x^* \in X$ and $V_{02} \in \mathbb{R}$ for which*

$$Dx^* = S^{V_{02}} x^*.$$

Moreover, $l(x^*) \le 2r_U$.

Proof: From Lemma 9, Lemma 11, Lemma 6 and Lemma 12, the map

$$\phi = \psi \circ \widehat{D} \circ \psi^{-1} : [0, 4r_U] \to [0, 4r_U]$$

is continuous.

From the Fixed Point Theorem, there is $c \in [0, 4r_U]$ such that $\phi(c) = c$. Hence $\widehat{D}\psi^{-1}(c) = \psi^{-1}(c)$, *i.e.*, if $x^* \in X$ and $l(x^*) = c$, then there is $V_{02} \in \mathbb{R}$ such that $Dx^* = S^{V_{02}}x^*$.

Suppose that $l(x^*) > 2r_U$. Then, from Lemma 6,

$$l(Dx^*) = l(x^*) + (V_{12} - V_{01})\,.$$

Since $V_{01} > V_{12}$, then

$$l(Dx^*) < l(x^*)\,.$$

It is a contradiction.

Lemma 13 is proved.

Lemma 14 *If $V_{01} > V_{12}$, then there are $x^* \in X$ and $V_{02} \in \mathbb{R}$ for which*

$$\mathcal{L}(D^t x^*) = \mathcal{L}(x^*) + tV_{02} \quad \textit{and} \quad \mathcal{R}(D^t x^*) = \mathcal{R}(x^*) + tV_{02} \quad \textit{for all} \quad t \in \mathbb{Z}_+$$

and $V_{02} = L_{02} = R_{02}$.

Proof: From Lemma 13, there are $x^* \in X$ and $V_{02} \in \mathbb{R}$ for which $Dx^* = S^{V_{02}}x^*$. Then, from Lemma 4,

$$\mathcal{L}(D^t x^*) = \mathcal{L}(x^*) + tV_{02} \quad \text{and} \quad \mathcal{R}(D^t x^*) = \mathcal{R}(x^*) + tV_{02} \quad \text{for all} \quad t \in \mathbb{Z}_+\,.$$

So, let us show that $V_{02} = L_{02} = R_{20}$.

There are $q, q' \in \mathbb{R}$ such that

$$S^q x^* \prec j_{02} \prec S^{q'} x^*\,.$$

Therefore from monotonicity, shift-invariance, Lemma 4 and Lemma 3

$$\frac{q + \mathcal{L}(x^*) + tV_{02}}{t} \geq \frac{\mathcal{L}(D^t j_{02})}{t} \geq \frac{q' + \mathcal{L}(x^*) + tV_{02}}{t} \quad \text{for all} \quad t \in \mathbb{Z}_+\,.$$

So, from the Squeeze Theorem, $L_{02} = V_{02}$.

Similarly, one can prove that $R_{02} = V_{02}$.

Lemma 14 is proved.

Notice that Lemma 14 completes an alternative proof for the existence of L_{02} and R_{02}.

Proof of Theorem 2: Once we have defined the (01)-velocity and the (12)-velocity, our study can be partitioned in two cases, namely, $V_{01} \leq V_{12}$ and $V_{01} > V_{12}$.

If $V_{01} \leq V_{12}$, then result follows from Lemma 7 and Lemma 8.

If $V_{01} > V_{12}$, then the result follows from Lemma 14.

The proof for the left-continuous case is similar.

Theorem 2 is proved.

Lemma 15 *Let D be a one-dimensional regular operator. Then*

$$V_{12} \leq R_{02}\,.$$

Proof: If $V_{01} \leq V_{12}$, then $R_{02} = V_{12}$. So, suppose that $V_{01} > V_{12}$.
From Lemma 13, there is $x^* \in X$ with $\mathcal{R}(x^*) = 0$ and $V_{02} \in \mathbb{R}$ such that $Dx^* = S^{V_{02}}x^*$.
Notice that $x^* \prec j_{12}$.
From monotonicity

$$Dx^* \prec Dj_{12}\,,$$

whence

$$S^{V_{02}}x^* \prec S^{V_{12}}j_{12}\,.$$

From Lemma 3,

$$\mathcal{R}(S^{V_{02}}x^*) \geq \mathcal{R}(S^{V_{12}}j_{12})\,,$$

whence

$$R_{02} = V_{02} \geq V_{12}\,.$$

Lemma 15 is proved.

Lemma 15 is employed in Lemma 27.

2.3 Proof of Theorem 3

Proof of Theorem 3: From Lemma 13, there exist $x^* \in X$ with $\mathcal{R}(x^*) = 0$ and $V_{02} \in \mathbb{R}$ such that $Dx^* = S^{V_{02}}x^*$. Therefore

$$S^{l(x^*)}x^* \prec j_{02} \prec x^*$$

From monotonicity and shift-invariance

$$S^{l(x^*)}D^t x^* \prec D^t j_{02} \prec D^t x^* \quad \text{for all} \quad t \in \mathbb{Z}_+\,.$$

Hence from Lemma 4 and Lemma 3

$$-l(x^*) + tV_{02} = \mathcal{L}(D^t x^*) \leq \mathcal{L}(D^t j_{02}) \leq \mathcal{R}(D^t j_{02}) \leq l(x^*) + \mathcal{R}(D^t x^*) = l(x^*) + tV_{02}\,.$$

Since $l(x^*) \leq 2r_U$,

$$-2r_U + tV_{02} \leq \mathcal{L}(D^t j_{02}) \leq \mathcal{R}(D^t j_{02}) \leq 2r_U + tV_{02} \quad \text{for all} \quad t \in \mathbb{Z}_+\,.$$

Theorem 3 is proved.

2.4 Proof of Theorem 4

Lemma 16 *If $L_{20} \geq R_{02}$, then D is a non-2-degrader.*

Proof: From Theorem 2, there exist a right-continuous (02)-ladder x and a left-continuous (20)-ladder x' such that

$$\mathcal{R}(D^t x) = tR_{02} \quad \text{and} \quad \mathcal{L}(D^t x') = 2r_U + tL_{20} \quad \text{for all} \quad t \in \mathbb{Z}_+ . \tag{2.11}$$

Consider the island $\hat{x}$ given by

$$\hat{x}_p = \begin{cases} 2 & \text{if} \quad 0 \leq p \leq 2r_U, \\ 1 & \text{if} \quad -l(x) \leq p < 0 \quad \text{or} \quad 2r_U < p \leq 2r_U + l(x'), \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

Let us prove the following statements:

▷ $(D^t x)_p = (D^t \hat{x})_p$ for all $p \in (-\infty, \mathcal{L}(D^t x')]$,

▷ $(D^t x')_p = (D^t \hat{x})_p$ for all $p \in [\mathcal{R}(D^t x), \infty)$,

▷ $(D^t \hat{x})_p = 2$ for all $p \in [\mathcal{R}(D^t x), \mathcal{L}(D^t x')]$ for all $t \in \mathbb{Z}_+$

Indeed, they are true for $t = 0$. See Figure 2.5.

Suppose that they are true for an arbitrary $t > 0$. So, from the inductive hypothesis,

$$(D^t x)_p = (D^t \hat{x})_p \quad \text{for all} \quad p \in (-\infty, \mathcal{L}(D^t x')]$$

and

$$(D^t x')_p = (D^t \hat{x})_p \quad \text{for all} \quad p \in [\mathcal{R}(D^t x), \infty) .$$

Therefore, from (1.4),

$$(D^{t+1} x)_p = (D^{t+1} \hat{x})_p \quad \text{for all} \quad p \in (-\infty, \mathcal{L}(D^t x') - r_U] \tag{2.13}$$

and

$$(D^{t+1} x')_p = (D^{t+1} \hat{x})_p \quad \text{for all} \quad p \in [\mathcal{R}(D^t x) + r_U, \infty) . \tag{2.14}$$

Let us prove that $(D^{t+1} x)_p = (D^{t+1} \hat{x})_p$ for all $p \in (\mathcal{L}(D^t x') - r_U, \mathcal{L}(D^{t+1} x')]$.

From (2.11)

$$\mathcal{L}(D^t x') - \mathcal{R}(D^t x) = 2r_U + t(L_{20} - R_{02}) .$$

Therefore

$$\mathcal{L}(D^t x') - \mathcal{R}(D^t x) \geq 2 r_U \,. \tag{2.15}$$

So, from (2.15)

$$\mathcal{L}(D^t x') - r_U \geq \mathcal{R}(D^t x) + r_U \,. \tag{2.16}$$

Notice also that $\mathcal{R}(D^t x) + r_U \geq \mathcal{R}(D^{t+1} x)$.

Let p be an arbitrary element of $(\,\mathcal{L}(D^t x') - r_U, \mathcal{L}(D^{t+1} x')\,]$. From (2.16),

$$p \geq \mathcal{R}(D^t x) + r_U \geq \mathcal{R}(D^{t+1} x) \,.$$

So, $(D^{t+1} x)_p = 2$.

On other hand, since $p \leq \mathcal{L}(D^{t+1} x')$, then $(D^{t+1} x')_p = 2$. Hence, from (2.14)

$$(D^{t+1} x')_p = (D^{t+1} \hat{x})_p = 2 \,. \tag{2.17}$$

Therefore, from (2.13) and (2.17)

$$(D^{t+1} x)_p = (D^{t+1} \hat{x})_p \quad \text{for all} \quad p \in (\,-\infty, \mathcal{L}(D^{t+1} x')\,] \,.$$

Similarly, one can prove that

$$(D^{t+1} x')_p = (D^{t+1} \hat{x})_p \quad \text{for all} \quad p \in [\mathcal{R}(D^{t+1} x), \infty\,) \,.$$

Thus

$$(D^{t+1} \hat{x})_p = 2 \quad \text{for all} \quad p \in [\,\mathcal{R}(D^{t+1} x), \mathcal{L}(D^{t+1} x')\,] \,.$$

So they are true for $t+1$. The statements are proved.

Hence, island $\hat{x}$ is not 2-degraded by D.

Lemma 16 is proved.

Lemma 17 *If $R_{02} > L_{20}$, then D is a linear 2-degrader.*

Proof: Consider the island d_R defined in (1.7). From Theorem 2, there exist a right-continuous (02)-ladder x and a left-continuous (20)-ladder x' for which

$$\mathcal{R}(D^t x) = -R + t R_{02} \quad \text{and} \quad \mathcal{L}(D^t x') = R + t L_{20} \quad \text{for all} \quad t \in \mathbb{Z}_+ \,. \tag{2.18}$$

Notice, see Figure 2.6, that $x \succ d_R$ and $x' \succ d_R$.

At first, let us prove the following assertion:

If there is $\tau \in \mathbb{Z}_+$ for which $\mathcal{R}(D^\tau x) > \mathcal{L}(D^\tau x')$, then $max(D^\tau d_R) < 2$.

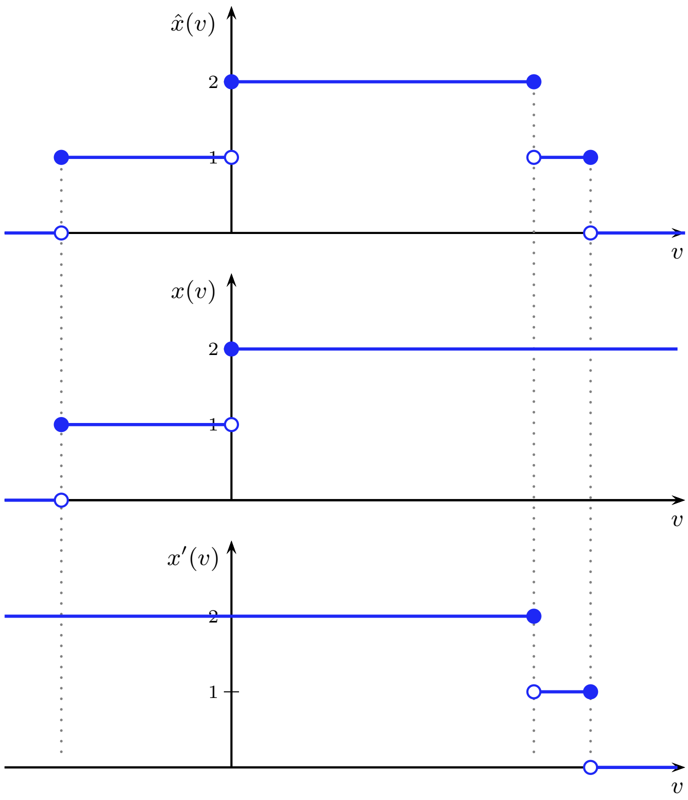

Figure 2.5: *Configurations $\hat{x}$, x and x'.*

Indeed, suppose for absurd that there is $p \in \mathbb{R}$ such that $(D^\tau d_R)_p = 2$. Since $x \succ d_R$ and $x' \succ d_R$, then from monotonicity

$$D^\tau x \succ D^\tau d_R \quad \text{and} \quad D^\tau x' \succ D^\tau d_R\,.$$

Hence the non-empty set $\{\, p \in \mathbb{R} \,|\, (D^\tau d_R)_p = 2\}$ is a subset of $\{\, p \in \mathbb{R} \,|\, (D^\tau x)_p = 2\}$ and also a subset of $\{\, p \in \mathbb{R} \,|\, (D^\tau x')_p = 2\}$. Therefore

$$\inf\{\, p \in \mathbb{R} \,|\, (D^\tau d_R)_p = 2\} \geq \inf\{\, p \in \mathbb{R} \,|\, (D^\tau x)_p = 2\} = \mathcal{R}(D^\tau x) \tag{2.19}$$

and

$$\mathcal{L}(D^\tau x') = \sup\{\, p \in \mathbb{R} \,|\, (D^\tau x')_p = 2\} \geq \sup\{\, p \in \mathbb{R} \,|\, (D^\tau d_R)_p = 2\}\,. \tag{2.20}$$

Since $\mathcal{R}(D^\tau x) > \mathcal{L}(D^\tau x')$, then from (2.21) and (2.20)

$$\inf\{\, p \in \mathbb{R} \,|\, (D^\tau d_R)_p = 2\} > \sup\{\, p \in \mathbb{R} \,|\, (D^\tau d_R)_p = 2\}\,.$$

It is a contradiction. Thus the assertion is proved.

From (2.18)

$$\mathcal{R}(D^t x) - \mathcal{L}(D^t x') = -2R + t(R_{02} - L_{20}) \quad \text{for all} \quad t \in \mathbb{Z}_+\,. \tag{2.21}$$

From division,

$$2R = t_0(R_{02} - L_{20}) + r_0$$

where $t_0 \in \mathbb{Z}_+$ and $0 \leq r_0 < (R_{02} - L_{20})$. Hence

$$0 \geq -r_0 = -2R + t_0(R_{02} - L_{20}) > -(R_{02} - L_{20}). \tag{2.22}$$

Therefore, from (2.22) and (2.21),

$$-r_0 + (R_{02} - L_{20}) = -2R + (t_0 + 1)(R_{02} - L_{20}) = \mathcal{R}(D^{t_0+1}x) - \mathcal{L}(D^{t_0+1}x') > 0.$$

Thus $max(D^{t_0+1}d_R) < 2$.

Notice that

$$t_0 + 1 = \frac{2R}{R_{02} - L_{20}} - \frac{r_0}{R_{02} - L_{20}} + 1 \leq \frac{2R}{R_{02} - L_{20}} + 1\,.$$

Hence

$$\tau_2^D(d_R) \leq (R_{02} - L_{20})^{-1}2R + 1\,.$$

Lemma 17 is proved.

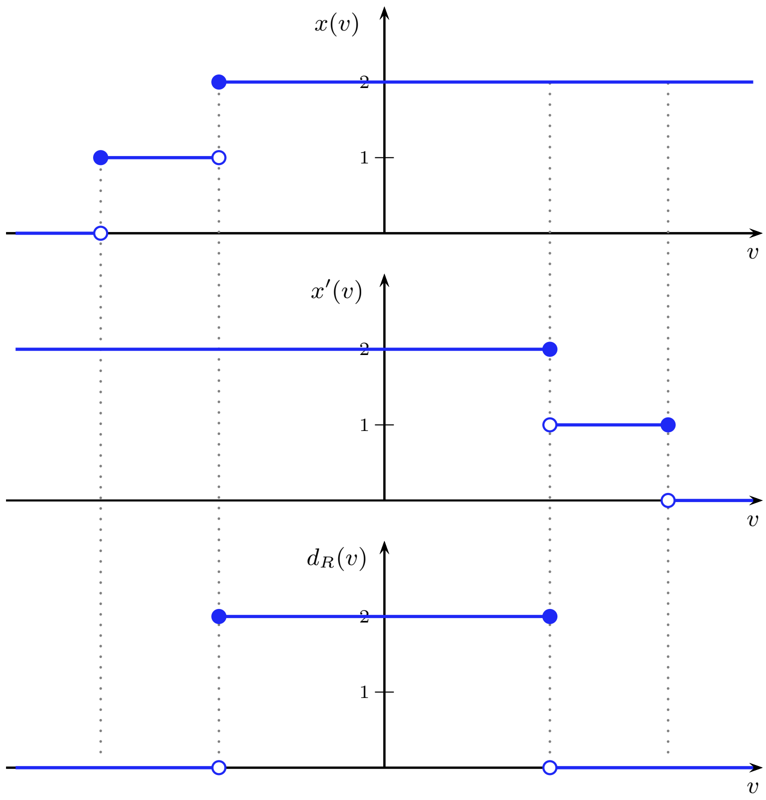

Figure 2.6: *Configurations x, x' and d_R.*

Proof of Theorem 4: It is a direct consequence of Lemma 16 and Lemma 17.

Theorem 4 is proved.

CHAPTER 3

Linear 2-degrader sufficient condition in $\mathbb{R}^2$

Having made some discovery, however modest, we should not fail to inquire whether there is something more behind it, we should not miss the possibilities opened up by the new result, we should try to use again the procedure used. Exploit your success!

— **George Polya**

LET us call by *direction* any vector δ in $\mathbb{R}^2$ such that $\|\delta\| = 1$, where $\| \ \|$ denotes the usual norm. Given a direction δ, a $\mathbb{R}^2$-configuration $y : \mathbb{R}^2 \to M$ is called a *δ-configuration* if

$$\forall v, w \in \mathbb{R}^2 \ : \ \langle v, \delta \rangle = \langle w, \delta \rangle \Longrightarrow y_v = y_w .$$

where $\langle \ , \ \rangle$ is the usual inner product.

From any δ-configuration $y : \mathbb{R}^2 \to M$ one can define a $\mathbb{R}$-configuration $y^\delta : \mathbb{R} \to M$ given by

$$y^\delta(p) = y(p \cdot \delta) \quad \text{for all} \quad p \in \mathbb{R}\,, \tag{3.1}$$

where the symbol $\cdot$ denotes the multiplication by scalar[1]. For instance, the configuration

[1] The multiplication by scalar $\cdot : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is given by $p \cdot (v_1, v_2) = (pv_1, pv_2)$.

$\tilde{x} : \mathbb{R}^2 \to M$ given by

$$\tilde{x}_v = \begin{cases} 0 & \text{if} \quad \langle v, (\sqrt{2}/2, \sqrt{2}/2)\rangle < -1, \\ 1 & \text{if} \quad -1 \leq \langle v, (\sqrt{2}/2, \sqrt{2}/2)\rangle < 1, \\ 2 & \text{otherwise,} \end{cases}$$

is a $(\sqrt{2}/2, \sqrt{2}/2)$-configuration. See Figure 3.1. Configuration $\tilde{x}^{(\sqrt{2}/2, \sqrt{2}/2)} : \mathbb{R} \to M$ is given by

$$\tilde{x}^{(\sqrt{2}/2, \sqrt{2}/2)}(p) = \begin{cases} 0 & \text{if} \quad p < -1, \\ 1 & \text{if} \quad -1 \leq p < 1, \\ 2 & \text{otherwise.} \end{cases}$$

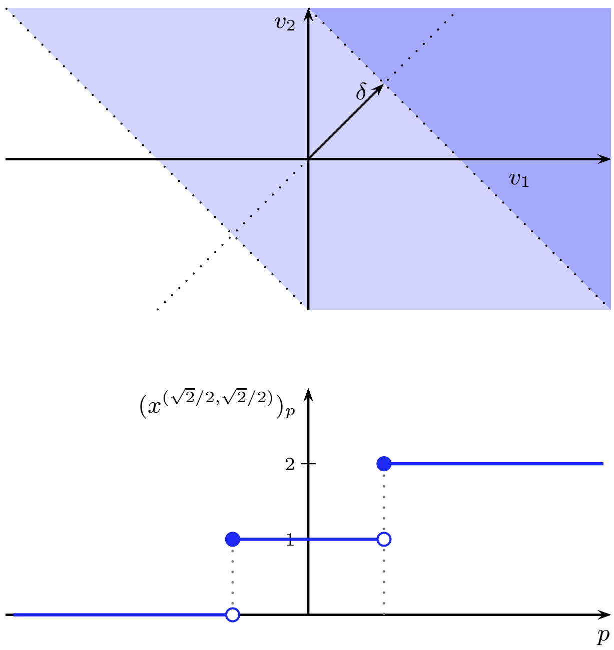

Figure 3.1: $(\sqrt{2}/2, \sqrt{2}/2)$*-configuration* $\tilde{x}$ *and configuration* $\tilde{x}^{(\sqrt{2}/2, \sqrt{2}/2)}$.

Let us define

$$U^\delta = \{u_j^\delta = \langle u_i, \delta\rangle \mid u_i \in U\}.$$

Notice that U^δ often has less elements than U. Let us denote the cardinality of U^δ by k_δ $(1 \leq k_\delta \leq k)$. For instance, consider the neighboorhood $U = \{(0,1), (0,0), (0,-1), (1,0)\}$. Then $U^{(0,1)} = \{1, 0, -1\}$, $U^{(1,0)} = \{0, 1\}$ and $U^{(\sqrt{2}/2, \sqrt{2}/2)} = \{\sqrt{2}/2, 0, -\sqrt{2}/2\}$.

The transition map $f^\delta : M^{k_\delta} \to M$ is defined by

$$f^\delta(a_1, \dots, a_{k_\delta}) = f(b_1, \dots, b_k)$$

where $b_i = a_j$ whenever $\langle u_i, \delta \rangle = u_j^\delta$. Notice that f^δ is monotonic and

$$f^\delta(a, \dots, a) = a \quad \text{for all} \quad a \in M\,.$$

For example, consider the neighboorhood $U = \{(0,1), (0,0), (0,-1), (1,0)\}$ and the transition map $f : M^4 \to M$ given by

$$f(a,b,c,d) = \begin{cases} 1 & \text{if} \quad b = 2,\ a \leq 1,\ d = 0, \\ 0 & \text{if} \quad b = 1,\ c = 0,\ d = 0, \\ b & \text{otherwise.} \end{cases}$$

Thus $f^{(0,1)} : M^3 \to M$ is given by $f^{(0,1)}(a,b,c) = b$, $f^{(1,0)} : M^2 \to M$ is given by $f^{(1,0)}(a,d) = a$ and $f^{(\sqrt{2}/2, \sqrt{2}/2)} : M^3 \to M$ is given by

$$f^{(\sqrt{2}/2,\sqrt{2}/2)}(a,b,c) = \begin{cases} 1 & \text{if} \quad a = 0, b = 2\,, \\ 0 & \text{if} \quad b = 1, a = 0, c = 0\,, \\ b & \text{otherwise.} \end{cases}$$

For all direction δ one can define a regular operator $D^\delta : M^{\mathbb{R}} \to M^{\mathbb{R}}$ by

$$(D^\delta x)_p = f^\delta(x_{p+u_1^\delta}, \dots, x_{p+u_{k_\delta}^\delta}) \quad \text{for all} \quad p \in \mathbb{R}\,.$$

For instance, if we consider the shift operator $S^w : M^{\mathbb{R}^2} \to M^{\mathbb{R}^2}$, then $(S^w)^\delta = S^{\langle w,\delta\rangle} : M^{\mathbb{R}} \to M^{\mathbb{R}}$. Indeed, the shift operator S^w has transition map $f : M \to M$ given by $f(a) = a$ and neighborhood $U = \{-w\}$. Hence $f^\delta : M \to M$ is given by $f^\delta(a) = a$ and $U^\delta = \{-\langle w, \delta \rangle\}$. Thus $((S^w)^\delta x)_p = f^\delta(x_{p-\langle w,\delta\rangle}) = x_{p-\langle w,\delta\rangle} = S^{\langle w,\delta\rangle}x$.

From Theorem 1, there are

$$\lim_{t\to\infty} \frac{\mathcal{L}((D^\delta)^t j_{02})}{t} = L_{02}(D^\delta)\,, \quad \lim_{t\to\infty} \frac{\mathcal{R}((D^\delta)^t j_{02})}{t} = R_{02}(D^\delta)\,,$$

$$\lim_{t\to\infty} \frac{\mathcal{L}((D^\delta)^t j_{20})}{t} = L_{20}(D^\delta) \quad \text{and} \quad \lim_{t\to\infty} \frac{\mathcal{R}((D^\delta)^t j_{20})}{t} = R_{20}(D^\delta)$$

which are the left (02)-velocity, right (02)-velocity, left (20)-velocity, and right (20)-velocity of D^δ respectively. Furthermore, from Lemma 1,

$$D^\delta j_{01} = S^{V_{01}(D^\delta)} j_{01} \quad \text{and} \quad D^\delta j_{12} = S^{V_{12}(D^\delta)} j_{12}$$

where $V_{01}(D^\delta)$ and $V_{12}(D^\delta)$denotes the (01)-velocity and the (12)-velocity of D^δ respectively.

For each direction δ, let us denote the following *closed half-space* by

$$H_\delta^D = \{v \in \mathbb{R}^2 \,|\, \langle \delta, v\rangle \geq R_{02}(D^\delta)\}.$$

Any H_δ^D is a non-empty closed and convex set. At last, we define also a closed and convex subset of $\mathbb{R}^2$ by

$$\sigma_D = \bigcap_\delta H_\delta^D\,.$$

Chapter 3 has just one Theorem, namely,

Theorem 5 *If $\sigma_D = \emptyset$, then*

- ▷ *either there is a direction δ for which $H_\delta^D \cap H_{-\delta}^D = \emptyset$*
- ▷ *or there are three directions $\delta_1, \delta_2, \delta_3$ for which $H_{\delta_1}^D \cap H_{\delta_2}^D \cap H_{\delta_3}^D = \emptyset$*

and in both cases the two-dimensional regular operator D is a linear 2-degrader.

Theorem 5 shows that $\sigma_D = \emptyset$ is a sufficient condition to be linear 2-degrader. Is it also necessary? The author guesses it is, but he can not prove it yet. Nevertheless, the results presented in Chapter 4 enhance his belief that D is not a linear 2-degrader whenever $\sigma_D \neq \emptyset$.

In section 3.1 some preliminary results are proved. Theorems 5 is proved in Section 3.2.

3.1 Preliminary results

Lemma 18 *Let D be a regular operator and w be a point in $\mathbb{R}^n$. Define the regular operator $\tilde{D} = S^w \circ D$. Then*

$$\tau_2^D(y) = \tau_2^{\tilde{D}}(y) \quad \textit{for all} \quad y \in M^{\mathbb{R}^n}\,.$$

Proof: At first, let us prove that

$$max(\tilde{D}^t y) = max(D^t y) \quad \text{for all} \quad t \in \mathbb{Z}_+\,. \tag{3.2}$$

By definition

$$max(Dy) = \max\{\,(Dy)_v \,|\, v \in \mathbb{R}^n\,\}\,.$$

Define $v' = v + w$, then

$$\max\{\,(Dy)_{v'-w} \,|\, v' \in \mathbb{R}^n\,\} = \max\{\,(S^w Dy)_{v'} \,|\, v' \in \mathbb{R}^n\,\} = max(\tilde{D}y)\,.$$

So it is true for $t = 1$.

Suppose that it is true for $t > 1$. Therefore from the inductive hypothesis

$$max(D^t(Dy)) = max(\tilde{D}^t(Dy)) \,.$$

From the shift-invariance

$$max(D(D^t x)) = max(D(\tilde{D}^t x)) \,,$$

whence from the case $t = 1$

$$max(D(\tilde{D}^t x)) = max(\tilde{D}(\tilde{D}^t x)) \,.$$

So it is true for $t + 1$.

Let y be an $\mathbb{R}^n$-configuration. If $\tau_2^D(y) = \infty$, then by definition $max(D^t y) = 2$ for all $t \in \mathbb{Z}_+$. From (3.2), $max(\tilde{D}^t y) = max(D^t y) = 2$ for all $t \in \mathbb{Z}_+$. Hence $\tau_2^{\tilde{D}} = \infty$. On other hand, if y is 2-degraded by D,

$$\tau_2^D(y) = \min\{t \in \mathbb{Z}_+ \,|\, max(D^t y) < 2\,\} = \min\{t \in \mathbb{Z}_+ \,|\, max(\tilde{D}^t y) < 2\,\} = \tau_2^{\tilde{D}}(y) \,.$$

Lemma 18 is proved.

LEMMA 19 *If y is a δ-configuration, then $D^t y$ is also a δ-configuration for all $t \in \mathbb{Z}_+$.*

PROOF: Suppose that y is a δ-configuration. Let us prove that Dy is also a δ-configuration. Let v and w be two points of $\mathbb{R}^2$ such that $\langle v, \delta \rangle = \langle w, \delta \rangle$. Then

$$\langle v + u_i, \delta \rangle = \langle w + u_i, \delta \rangle \quad \text{for all} \quad i \in \{1, \ldots, k\} \,. \tag{3.3}$$

Since y is a δ-configuration and from (3.3)

$$y_{v+u_i} = y_{w+u_i} \quad \text{for all} \quad i \in \{1, \ldots, k\} \,. \tag{3.4}$$

From (1.4) and (3.4),

$$(Dy)_v = f(y_{w+u_1}, \ldots, y_{w+u_k}) = (Dy)_w.$$

Thus it is true for $t = 1$.

Suppose that it is true for an arbitrary natural $t > 1$, i.e., $D^t y$ is a δ-configuration. Hence, from the case for $t = 1$,

$$(D^{t+1} y)_v = (D(D^t y))_v = (D(D^t y))_w = (D^{t+1} y)_w \,.$$

Thus it is true for $t + 1$.

Lemma 19 is proved.

Lemma 20 *Let y be a δ-configuration. Then*

$$(D^t y)_v = ((D^\delta)^t y^\delta)_{\langle v,\delta\rangle} \quad \textit{for all} \quad v \in \mathbb{R}^2\,, \quad t \in \mathbb{Z}_+\,.$$

Moreover, $\tau_2^D(y) = \tau_2^{D^\delta}(y^\delta)$.

Proof: Let $\{\delta,\, \delta^\perp\}$ be an orthonormal basis of $\mathbb{R}^2$ and y be a δ-configuration. By definition of the regular operator D,

$$(Dy)_v = f(y_{v+u_1}, \ldots, y_{v+u_k}) \quad \text{for all} \quad v \in \mathbb{R}^2\,.$$

Any point $v \in \mathbb{R}^2$ can be written as

$$v = \langle v, \delta\rangle\delta + \langle v, \delta^\perp\rangle\delta^\perp,$$

then

$$(Dy)_v = f(y_{\langle v+u_1,\delta\rangle\delta+\langle v+u_1,\delta^\perp\rangle\delta^\perp}, \ldots, y_{\langle v+u_n,\delta\rangle\delta+\langle v+u_k,\delta^\perp\rangle\delta^\perp}).$$

Since y is δ-configuration

$$(Dy)_v = f(y_{\langle v,\delta\rangle\delta+\langle u_1,\delta\rangle\delta}, \ldots, y_{\langle v,\delta\rangle\delta+\langle u_k,\delta\rangle\delta})\,. \tag{3.5}$$

From the definition of U^δ and f^δ and from (3.5)

$$(Dy)_v = f^\delta(y_{(\langle v,\delta\rangle+u_1^\delta)\delta}, \ldots, y_{(\langle v,\delta\rangle+u_{k_\delta}^\delta)\delta}) \tag{3.6}$$

and from definition of y^δ, (3.1),

$$(Dy)_v = f^\delta(y^\delta_{(\langle v,\delta\rangle+u_1^\delta)}, \ldots, y^\delta_{(\langle v,\delta\rangle+u_{k_\delta}^\delta)}) = (D^\delta y^\delta)_{\langle v,\delta\rangle}\,.$$

So it is true for $t = 1$.

Suppose that it is true for an arbitrary $t > 1$. From Lemma 19, $D^t y$ is a δ-configuration. Hence from (3.6)

$$(D^{t+1}y)_v = (D(D^t y))_v = f^\delta((D^t y)_{(\langle v,\delta\rangle+u_1^\delta)\delta}, \ldots, (D^t y)_{(\langle v,\delta\rangle+u_{k_\delta}^\delta)\delta})\,.$$

From the inductive hypothesis

$$(D^t y)_{(\langle v,\delta\rangle+u_j^\delta)\delta} = ((D^\delta)^t y^\delta)_{\langle v,\delta\rangle+u_j^\delta} \quad \text{for all} \quad j \in \{1, \ldots, k_\delta\}\,.$$

Therefore

$$(D^{t+1}y)_v = f^\delta(((D^\delta)^t y^\delta)_{\langle v,\delta\rangle+u_1^\delta}, \ldots, ((D^\delta)^t y^\delta)_{\langle v,\delta\rangle+u_{k_\delta}^\delta}) = ((D^\delta)^{t+1} y^\delta)_{\langle v,\delta\rangle}.$$

Thus it is proved for $t+1$.

Let y be an δ-configuration. Since $(D^t y)_v = ((D^\delta)^t y^\delta)_{\langle v,\delta\rangle}$ for all $v \in \mathbb{R}^2$ and $t \in \mathbb{Z}_+$, then

$$max(D^t y) = max((D^\delta)^t y^\delta).$$

If $\tau_2^D(y) = \infty$, then by definition $max(D^t y) = 2$ for all $t \in \mathbb{Z}_+$. Hence

$$max(D^t y) = max((D^\delta)^t y^\delta) = 2 \quad \text{for all} \quad t \in \mathbb{Z}_+.$$

Thus $\tau_2^{D^\delta} = \infty$. On other hand, if y is 2-degraded by D,

$$\tau_2^D(y) = \min\{t \in \mathbb{Z}_+ \,|\, max(D^t y) < 2\} = \min\{t \in \mathbb{Z}_+ \,|\, max((D^\delta)^t y^\delta) < 2\} = \tau_2^{D^\delta}(y^\delta)\,.$$

Lemma 20 is proved.

LEMMA 21 *Let D be a regular operator and δ be an arbitrary direction. Then*

$$R_{02}(D^{-\delta}) = -L_{20}(D^\delta)\,.$$

PROOF: At first, let us prove that

$$((D^{-\delta})^t j_{02})_p = ((D^\delta)^t j_{20})_{-p} \quad \text{for all} \quad p \in \mathbb{R}\,. \tag{3.7}$$

Notice that it is true for $t = 0$, i.e.,

$$(j_{02})_p = (j_{20})_{-p} \quad \text{for all} \quad p \in \mathbb{R}\,.$$

Suppose that it is true for an arbitrary $t > 0$. Let p be an arbitrary element of $\mathbb{R}$. By definition (1.4)

$$((D^{-\delta})^{t+1} j_{02})_p = f^{-\delta}(((D^{-\delta})^t j_{02})_{p+u_1^{-\delta}}, \ldots, ((D^{-\delta})^t j_{02})_{p+u_{k_{-\delta}}^{-\delta}})\,.$$

Notice that

$$k_\delta = k_{-\delta}\,, \quad u_j^\delta = -u_j^{-\delta} \quad \text{for all} \quad j \in \{1, 2, \ldots, k_\delta\}\,. \tag{3.8}$$

and

$$f^{-\delta} = f^\delta\,. \tag{3.9}$$

From the inductive hypothesis

$$((D^{-\delta})^t j_{02})_{p+u_j^{-\delta}} = ((D^\delta)^t j_{20})_{-p-u_j^{-\delta}} \quad \text{for all} \quad j \in \{1, 2, \ldots, k_{-\delta}\}\,. \tag{3.10}$$

Therefore from (3.8), (3.9) and (3.10)

$$((D^{-\delta})^{t+1} j_{02})_p = f^\delta(((D^\delta)^t j_{20})_{-p+u_1^\delta}, \ldots, ((D^\delta)^t j_{20})_{-p+u_{k_\delta}^\delta}) = ((D^\delta)^{t+1} j_{20})_{-p}\,.$$

Thus it is true for $t+1$.

From (3.7),

$$\{\,-p \in \mathbb{R}\,|\,((D^{\delta})^t j_{20})_p = 2\,\} = \{\,p \in \mathbb{R}\,|\,((D^{-\delta})^t j_{02})_p = 2\,\}\,. \tag{3.11}$$

Now, by definition

$$\begin{aligned}-L_{20}(D^{\delta}) &= -\lim_{t\to\infty}\frac{\mathcal{L}((D^{\delta})^t j_{20})}{t} = \lim_{t\to\infty}\frac{-\mathcal{L}((D^{\delta})^t j_{20})}{t}\\ &= \lim_{t\to\infty}\frac{-\sup\{\,p\in\mathbb{R}\,|\,((D^{\delta})^t j_{20})_p = 2\,\}}{t}\\ &= \lim_{t\to\infty}\frac{\inf\{\,-p\in\mathbb{R}\,|\,((D^{\delta})^t j_{20})_p = 2\,\}}{t}\end{aligned}$$

From (3.11)

$$-L_{20}(D^{\delta}) = \lim_{t\to\infty}\frac{\inf\{\,p\in\mathbb{R}\,|\,((D^{-\delta})^t j_{02})_p = 2\,\}}{t} = \lim_{t\to\infty}\frac{\mathcal{R}((D^{-\delta})^t j_{02})}{t} = R_{02}(D^{-\delta})\,.$$

Lemma 21 is proved.

Lemma 22 *Let D be a regular operator and w be a point in $\mathbb{R}^2$. Define $\tilde{D} = S^w \circ D$. Then*

$$R_{02}(\tilde{D}^{\delta}) = \langle w,\delta\rangle + R_{02}(D^{\delta})\,.$$

Proof: At first, let us prove that

$$\tilde{D}^{\delta} = S^{\langle w,\delta\rangle}\circ D^{\delta}\,.$$

The operator $\tilde{D} : M^{\mathbb{R}^2} \to M^{\mathbb{R}^2}$ is given by

$$(\tilde{D}y)_v = (S^w \circ Dy)_v = (Dy)_{v-w} = f(y_{v-w+u_1},\ldots,y_{v-w+u_k})\,.$$

Notice that the transition map of $\tilde{D}$ is the same of D, namely, f. However, the neightborhooh of $\tilde{D}$ is

$$\tilde{U} = \{\,-w+u_1,\ldots,-w+u_k\,\}\,.$$

Therefore

$$\tilde{U}^{\delta} = \{\,-\langle w,\delta\rangle + u_1^{\delta},\ldots,-\langle w,\delta\rangle + u_{k_\delta}^{\delta}\,\}\,.$$

Thus

$$(\tilde{D}^{\delta}x)_p = f^{\delta}(x_{p-\langle w,\delta\rangle+u_1^{\delta}},\ldots,x_{p-\langle w,\delta\rangle+u_{k_\delta}^{\delta}}) = (D^{\delta}x)_{p-\langle w,\delta\rangle} = (S^{\langle w,\delta\rangle}\circ D^{\delta}x)_p\,.$$

Hence

$$R_{02}(\tilde{D}^{\delta}) = R_{02}(S^{\langle w,\delta\rangle}\circ D^{\delta})\,.$$

From Lemma 5

$$R_{02}(\tilde{D}^{\delta}) = \langle w, \delta \rangle + R_{02}(D^{\delta})\,.$$

Lemma 22 is proved.

3.2 Proof of Theorem 5

Let C be a non-empty convex set of $\mathbb{R}^n$. We say that δ is a *direction of recession* of C if $x + \lambda\delta \in C$ for all $\lambda \in \mathbb{R}_+$ and for all $x \in C$. The following result, which can be found at Rockafellar (1970), will be useful in proving Theorem 5.

Helly's Theorem: *Let $\{C_i\}_{i \in I}$ be a collection of non-empty closed convex sets in $\mathbb{R}^n$, where I is an arbitrary index set. Assume that for every direction δ there is $i \in I$ for which δ is not a direction of recession of C_i. If every subcollection of $n+1$ or fewer sets has non-empty intersection, then the entire collection has a non-empty intersection.*

Lemma 23 *Let D be a regular operator and δ be an arbitrary direction. Then*

$$H_{\delta}^{D} \cap H_{-\delta}^{D} = \emptyset \quad \Leftrightarrow \quad R_{02}(D^{\delta}) > L_{20}(D^{\delta})\,. \tag{3.12}$$

Proof: Suppose that $H_{\delta}^{D} \cap H_{-\delta}^{D} = \emptyset$. Consider $R_{02}(D^{\delta}) \cdot \delta$ and notice that $\langle R_{02}(D^{\delta}) \cdot \delta, \delta \rangle = R_{02}(D^{\delta})$. Therefore, by definition of H_{δ}^{D}, $R_{02}(D^{\delta}) \cdot \delta \in H_{\delta}^{D}$.
Since $H_{\delta}^{D} \cap H_{-\delta}^{D} = \emptyset$,

$$\langle R_{02}(D^{\delta}) \cdot \delta, -\delta \rangle < R_{02}(D^{-\delta}). \tag{3.13}$$

From Lemma 21 and (3.13), $R_{02}(D^{\delta}) > L_{20}(D^{\delta})$.
Conversely, suppose that $R_{02}(D^{\delta}) > L_{20}(D^{\delta})$. Let v be an arbitrary element of H_{δ}^{D}, i.e., $\langle v, \delta \rangle \geq R_{02}(D^{\delta})$. So, we have $\langle v, \delta \rangle > L_{20}(D^{\delta}) = -R_{02}(D^{-\delta})$ or equivalently $\langle v, -\delta \rangle < R_{02}(D^{-\delta})$. Thus $v \notin H_{-\delta}^{D}$. Hence $H_{\delta}^{D} \cap H_{-\delta}^{D} = \emptyset$.

Lemma 23 is proved.

Lemma 24 *If there is a direction δ for which $R_{02}(D^{\delta}) > L_{20}(D^{\delta})$, then D is a linear 2-degrader.*

Proof: Consider the disk d_R defined in (1.7) and $z_R : \mathbb{R}^2 \to M$ given by

$$z_R(v) = \begin{cases} 2 & \text{if} \quad -R \leq \langle v, \delta \rangle \leq R \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $d_R \prec z_R$ and $z_R^\delta : \mathbb{R} \to M$ is given by

$$z_R^\delta(p) = \begin{cases} 2 & \text{if} \quad -R \le p \le R \\ 0 & \text{otherwise.} \end{cases}$$

Since $R_{02}(D^\delta) > L_{20}(D^\delta)$, then from Lemma 17

$$\tau_2^{D^\delta}(z_R^\delta) \le \lambda_{D^\delta} R + 1\,.$$

From Lemma 20,

$$\tau_2^D(z_R) = \tau_2^{D^\delta}(z_R^\delta)\,.$$

At last, since $d_R \prec z_R$,

$$\tau_2^D(d_R) \le \tau_2^D(z_R) \le \lambda_{D^\delta} R + 1\,.$$

Lemma 24 is proved.

One may say that condition $H_\delta^D \cap H_{-\delta}^D = \emptyset$ resembles Toom's point of view in the sense that it concerns an intersection of convex sets. On other hand, $R_{02}(D^\delta) > L_{20}(D^\delta)$ is more like Galperin's approach since it is given in terms of a generalization of Galperin's velocities. Maybe equivalence (3.12) made the author for the first time realize that the set σ_P defined in Toom (1976) as an intersection of minimal zero-sets should be generalized in terms of closed semi-planes depending on some kind of velocities à la Galperin. Probably this generalization step have already been clear for Toom (maybe years) before he gave the author this research subject.

Lemma 25 *Let D be a regular operator. Assume that $H_\delta^D \cap H_{-\delta}^D \ne \emptyset$ for all δ, but there are three distinct directions $\delta_1, \delta_2, \delta_3$ for which*

$$H_{\delta_1}^D \cap H_{\delta_2}^D \cap H_{\delta_3}^D = \emptyset\,.$$

Then there is $w \in \mathbb{R}^2$ such that the operator $\tilde{D} = S^w \circ D$ has

$$R_{02}(\tilde{D}^{\delta_1}) = R_{02}(\tilde{D}^{\delta_2}) = R_{02}(\tilde{D}^{\delta_3}) > 0 \quad \textit{and} \quad H_{\delta_1}^{\tilde{D}} \cap H_{\delta_2}^{\tilde{D}} \cap H_{\delta_3}^{\tilde{D}} = \emptyset\,. \tag{3.14}$$

Proof: Since $H_{\delta_1}^D \cap H_{\delta_2}^D \cap H_{\delta_3}^D = \emptyset$, then $(H_{\delta_1}^D \cup H_{\delta_2}^D \cup H_{\delta_3}^D)^c$ corresponds to an open triangular region in $\mathbb{R}^2$. See Figure 3.2.

Let us denote by v_c the incenter of this region. Let i be an arbitrary element of $\{1, 2, 3\}$. From Lemma 22 operator $\tilde{D} = S^{-v_c} \circ D$ has

$$R_{02}(\tilde{D}^{\delta_i}) = R_{02}(D^{\delta_i}) - \langle v_c, \delta_i \rangle\,. \tag{3.15}$$

Since v_c is the incenter, then $R_{02}(D^{\delta_i}) - \langle v_c, \delta_i \rangle = r$ where r is the radius of the incircle.
Now, let us prove $H_{\delta_1}^{\tilde{D}} \cap H_{\delta_2}^{\tilde{D}} \cap H_{\delta_3}^{\tilde{D}} = \emptyset$.
Indeed, suppose for absurd that there is $v \in \mathbb{R}^2$ such that

$$\langle v, \delta_i \rangle \geq R_{02}(\tilde{D}^{\delta_i}) \quad \text{for all} \quad i \in \{1, 2, 3\}\,.$$

So, from (3.15),

$$\langle v + v_c, \delta_i \rangle \geq R_{02}(D^{\delta_i}) \quad \text{for all} \quad i \in \{1, 2, 3\}\,.$$

It is a contradiction because $H_{\delta_1}^{D} \cap H_{\delta_2}^{D} \cap H_{\delta_3}^{D} = \emptyset$.

Lemma 25 is proved.

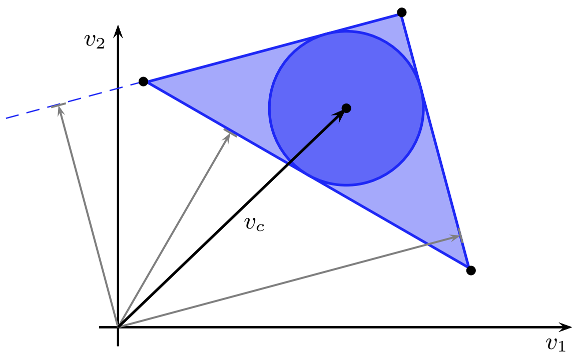

Figure 3.2: $(H_{\delta_1}^{D} \cup H_{\delta_2}^{D} \cup H_{\delta_3}^{D})^c$

LEMMA 26 *A regular operator $\tilde{D}$ satisfying (3.14) is a linear 2-degrader.*

PROOF: Consider the disk d_R defined in (1.7). Let i be an arbitrary element of $\{1, 2, 3\}$. Let $z_i : \mathbb{R}^2 \to M$ be a δ_i-configuration given by

$$z_i(v) = \begin{cases} 2 & \text{if} \quad \langle \delta_i, v \rangle \geq -R \\ 0 & \text{otherwise.} \end{cases}$$

Notice that

$$d_R \prec z_i\,. \tag{3.16}$$

Configuration $z_i^{\delta_i} : \mathbb{R} \to M$ is given by

$$z_i^{\delta_i}(p) = \begin{cases} 2 & \text{if} \quad p \geq -R \\ 0 & \text{otherwise.} \end{cases}$$

From Theorem 2, there exists a right-continuous (02)-ladder $z_i^* : \mathbb{R} \to M$ such that

$$\mathcal{R}((\tilde{D}^{\delta_i})^t z_i^*) = -R + tR_{02}(\tilde{D}^{\delta_i}) \quad \text{for all} \quad t \in \mathbb{Z}_+ \,. \tag{3.17}$$

Notice that

$$z_i^{\delta_i} \prec z_i^* \,. \tag{3.18}$$

From division, there are $t_0 \in \mathbb{Z}_+$ and $r_0 \in \mathbb{R}_+$ such that

$$0 \leq r_0 < R_{02}(\tilde{D}^{\delta_i}) \tag{3.19}$$

and

$$R = t_0 R_{02}(\tilde{D}^{\delta_i}) + r_0 \,. \tag{3.20}$$

From (3.17)

$$\mathcal{R}((\tilde{D}^{\delta_i})^{t_0+1} z_i^*) = -R + (t_0+1)R_{02}(\tilde{D}^{\delta_1}) \,. \tag{3.21}$$

From (3.20)

$$-R + t_0 R_{02}(\tilde{D}^{\delta_i}) = -r_0 \,,$$

whence

$$-R + (t_0+1)R_{02}(\tilde{D}^{\delta_i}) = -r_0 + R_{02}(\tilde{D}^{\delta_i}) \,. \tag{3.22}$$

So, from (3.21), (3.22) and (3.19)

$$\mathcal{R}((\tilde{D}^{\delta_i})^{t_0+1} z_i^*) = -r_0 + R_{02}(\tilde{D}^{\delta_i}) > 0 \,. \tag{3.23}$$

Let us prove that $max(D^{t_0+1} d_R) < 2$.

Indeed, suppose for absurd that there is $v_0 \in \mathbb{R}^2$ such that $(\tilde{D}^{t_0+1} d_R)_{v_0} = 2$. From (3.16) and monotonicity,

$$\tilde{D}^{t_0+1} d_R \prec \tilde{D}^{t_0+1} z_i \,,$$

whence

$$(\tilde{D}^{t_0+1} z_i)_{v_0} = 2 \,.$$

From Lemma 20

$$(\tilde{D}^{t_0+1} z_i)_{v_0} = ((\tilde{D}^{\delta_i})^{t_0+1} z_i^{\delta_i})_{\langle v_0, \delta_1 \rangle} = 2.$$

Since $z_i^{\delta_i} \prec z_i^*$, then $((\tilde{D}^{\delta_i})^{t_0+1} z_i^*)_{\langle v_0, \delta_1 \rangle} = 2$.

Since $\mathcal{R}((\tilde{D}^{\delta_i})^{t_0+1} z_i^*) = -r_0 + R_{02}(\tilde{D}^{\delta_i})$, then

$$\langle v_0, \delta_i \rangle \geq -r_0 + R_{02}(\tilde{D}^{\delta_i}) \,.$$

Let us denote

$$\lambda = \frac{R_{02}(\tilde{D}^{\delta_i})}{-r_0 + R_{02}(\tilde{D}^{\delta_i})}.$$

Therefore

$$\langle \lambda v_0, \delta_i \rangle \geq R_{02}(\tilde{D}^{\delta_i}).$$

Since i is arbitrary,

$$\lambda v_0 \in H_{\delta_1}^{\tilde{D}} \cap H_{\delta_2}^{\tilde{D}} \cap H_{\delta_3}^{\tilde{D}}.$$

It is a contradiction, because $H_{\delta_1}^{\tilde{D}} \cap H_{\delta_2}^{\tilde{D}} \cap H_{\delta_3}^{\tilde{D}} = \emptyset$. Thus $max(\tilde{D}^{t_0+1} d_R) < 2$.

From (3.20)

$$t_0 + 1 = \frac{R}{R_{02}(\tilde{D}^{\delta_1})} - \frac{r_0}{R_{02}(\tilde{D}^{\delta_1})} + 1 \leq \frac{R}{R_{02}(\tilde{D}^{\delta_1})} + 1.$$

Hence

$$\tau_2^{\tilde{D}}(d_R) \leq t_0 + 1 \leq (R_{02}(\tilde{D}^{\delta_1}))^{-1} R + 1.$$

Lemma 26 is proved.

PROOF OF THEOREM 5: Suppose that $\sigma_D = \emptyset$. Then, from Helly's Theorem,

▷ either there is a direction δ such that $H_{\delta}^{D} \cap H_{-\delta}^{D} = \emptyset$

▷ or there are three directions $\delta_1, \delta_2, \delta_3$ such that $H_{\delta_1}^{D} \cap H_{\delta_2}^{D} \cap H_{\delta_3}^{D} = \emptyset$.

If there is a direction δ such that $H_{\delta}^{D} \cap H_{-\delta}^{D} = \emptyset$, then, from Lemma 23, $R_{02}(D^{\delta}) > L_{20}(D^{\delta})$. Thus, from Lemma 24, D is a linear 2-degrader.

Suppose that $H_{\delta}^{D} \cap H_{-\delta}^{D} \neq \emptyset$ for any direction δ. Since there are three directions such that $H_{\delta_1}^{D} \cap H_{\delta_2}^{D} \cap H_{\delta_3}^{D} = \emptyset$, then, from Lemma 25, there is $w \in \mathbb{R}^2$ such that $\tilde{D} = S^w \circ D$ has $R_{02}(\tilde{D}^{\delta_1}) = R_{02}(\tilde{D}^{\delta_2}) = R_{02}(\tilde{D}^{\delta_3}) > 0$ and $H_{\delta_1}^{\tilde{D}} \cap H_{\delta_2}^{\tilde{D}} \cap H_{\delta_3}^{\tilde{D}} = \emptyset$. Hence, from Lemma 26, $\tilde{D}$ is a linear 2-degrader. Thus, from Lemma 18, D is also a linear 2-degrader.

Theorem 5 is proved.

CHAPTER 4

Non-linear 2-degrader conditions in $\mathbb{R}^2$

We have here a pattern of plausible inference:

$$\frac{\begin{array}{c}\textit{A implies B}\\ \textit{B true}\end{array}}{\textit{A more credible}}$$

The horizontal line again stands for "therefore". We shall call this pattern the fundamental inductive pattern or, somewhat shorter, the "inductive pattern".
This inductive pattern says nothing surprising. On the contrary, it express a belief which no reasonable person seems to doubt: the verification of a consequence renders a conjecture more credible. With a little attention, we can observe countless reasoning in everyday life, in the law courts, in science, etc., which appear to conform to our pattern.

— George Polya

In Theorem 4 we have seen that an one-dimensional 2-degrader D is necessarily a linear 2-degrader. Therefore, it is natural to wonder whether it is also true for a two-dimensional 2-degrader. Actually, it is not. A two-dimensional regular operator can be a 2-degrader and not be a linear 2-degrader.

For example, consider the neighboorhood $U = \{(0,1),(0,0),(0,-1),(1,0)\}$ and the transition map $f : M^4 \to M$ given by

$$f(a,b,c,d) = \begin{cases} 1 & \text{if} \quad b=2,\ a \leq 1,\ d=0, \\ 0 & \text{if} \quad b=1,\ c=0,\ d=0, \\ b & \text{otherwise.} \end{cases}$$

The regular operator D given by

$$(Dy)_v = f(y_{v+(0,1)}, y_{v+(0,0)}, y_{v+(0,-1)}, y_{v+(1,0)}) \quad \text{for all} \quad v \in \mathbb{R}^2$$

is a 2-degrader but it is not a linear 2-degrader one. Indeed, let y be a configuration given by

$$y_v = \begin{cases} 2 & \text{if} \quad \max\{|v_1|,|v_2|\} \leq n \\ 0 & \text{otherwise,} \end{cases}$$

where $n \in \mathbb{Z}_+$. Then $\tau_2^D(y) = 2n^2$. Observe that this transition map is the same presented in Lima de Menezes & Toom (2006).

Now we are ready to come back to the starting point of this research. Once, Professor Andrei Toom presented the following conjecture to the author:

Toom's Conjecture *Let D be a two-dimensional regular operator. If $0 \in \sigma_D$, then there is a positive real number C such that*

$$\tau_2^D(d_R) \geq CR^2 \quad \text{for all} \quad R \in \mathbb{R}_+ .$$

Actually, at that time, Professor Andrei Toom did not formulate such a precise conjecture. In fact, $R_{02}(D^\delta)$ had not been even defined yet. However, in our current definitions, that was what he meant. The author bets that the example presented in Lima de Menezes & Toom (2006) led to Toom's Conjecture, since it is not a linear 2-degrader and $\sigma_D = \{0\}$.

An island $c_{R,R'} : \mathbb{R}^2 \to M$ is called a *wedding cake* if it is given by

$$c_{R,R'}(v) = \begin{cases} 2 & \text{if} \quad \|v\| \leq R, \\ 1 & \text{if} \quad R < \|v\| \leq R', \\ 0 & \text{otherwise} \end{cases}$$

where $0 < R \leq R'$. Let A be a non-empty subset of $\mathbb{R}^2$. As usual, the number

$$diam(A) = \sup\{\, \|v-w\| \mid v,w \in A \,\}$$

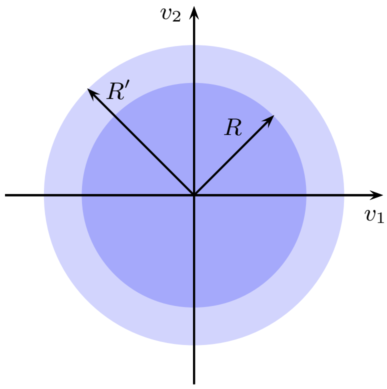

Figure 4.1: *Wedding cake* $c_{R,R'}$.

is called the *diameter* of A. Right now we do not know how to prove (or disprove) Toom's Conjecture. Nevertheless, an easier result which is similar to Toom's Conjecture is proved here, namely,

Theorem 6 *Suppose that* D *is a 2-degrader. If* $0 \in \sigma_D$ *and* $V_{01}(D^\delta) \leq 0$ *for any direction* δ*, then there is a positive real* C *such that*

$$\tau_2^D(c_{R,R+r_U}) \geq CR^2 \quad \textit{for all} \quad R > r_U\,.$$

It is a favorable sign for Toom's Conjecture.

Trying to prove Theorem 6, the author realized that it is plausible to formulate another Conjecture:

Conjecture 1 *If* $0 \in int(\sigma_D)$*, then there is a positive real* R *such that*

$$\lim_{t\to\infty} diam\{\, v \in \mathbb{R}^2 \,|\, (D^t d_R)_v = 2\} = \infty\,.$$

Again he only had a favorable sign:

Theorem 7 *Suppose that* $0 \in int(\sigma_D)$ *and there is* $\epsilon > 0$ *such that* $0 > -\epsilon \geq V_{01}(D^\delta)$ *for all direction* δ*. Then there is* $R > 0$ *such that*

$$\lim_{t\to\infty} diam\{\, v \in \mathbb{R}^2 \,|\, (D^t c_{R,R+2r_U})_v = 2\} = \infty\,.$$

Theorems 6 and 7 are proved in Sections 4.1 and 4.2 respectively.

4.1 Proof of Theorem 6

Lemma 27 *Suppose that $R > r_U$, $R_{02}(D^\delta) \leq 0$ and $V_{01}(D^\delta) \leq 0$ for all direction δ. Then*

$$c_{\tilde{R},\tilde{R}+r_U} \prec Dc_{R,R+r_U}\,,$$

and $\tilde{R} = (R^2 - r_U^2)^{1/2}$.

Proof: Let δ be an arbitrary direction. Notice, from (1.4), that

$$(Dc_{R,R+r_U})_{-\tilde{R}\cdot\delta}$$

depends only on points of

$$B_{r_U}[-\tilde{R}\cdot\delta] = \{\, v \in \mathbb{R}^2 \mid \|v + \tilde{R}\cdot\delta\| \leq r_U \,\}$$

as represented in Figure 4.2.

Consider the δ-configuration $y : \mathbb{R}^2 \to M$ presented also in Figure 4.2 and given by

$$y_v = \begin{cases} 2 & \text{if} \quad \langle v, \delta\rangle \geq -\tilde{R}\,, \\ 1 & \text{otherwise.} \end{cases}$$

Notice, vide Figure 4.2, that

$$c_{R,R+r_U}(v) \geq y(v) \quad \text{for all} \quad v \in B_{r_U}[-\tilde{R}\cdot\delta]\,.$$

Therefore, from monotonicity,

$$(Dc_{R,R+r_U})_{-\tilde{R}\cdot\delta} \geq (Dy)_{-\tilde{R}\cdot\delta}\,. \tag{4.1}$$

Configuration $y^\delta : \mathbb{R} \to M$ is given by

$$y^\delta(p) = \begin{cases} 2 & \text{if} \quad p \geq -\tilde{R}\,, \\ 1 & \text{otherwise.} \end{cases}$$

From Lemma 1,

$$(D^\delta y^\delta)_p = (S^{V_{12}(D^\delta)} y^\delta)_p = \begin{cases} 2 & \text{if} \quad p \geq -\tilde{R} + V_{12}(D^\delta)\,, \\ 1 & \text{otherwise.} \end{cases}$$

From Lemma 15, $V_{12}(D^\delta) \leq R_{02}(D^\delta) \leq 0$. Hence

$$(D^\delta y^\delta)_{-\tilde{R}} = 2\,.$$

From Lemma 20,

$$(Dy)_{-\tilde{R}\cdot\delta} = (D^{\delta}y^{\delta})_{-\tilde{R}} = 2\,. \tag{4.2}$$

From (4.1) and (4.2)

$$(Dc_{R,R+r_U})_{-\tilde{R}\cdot\delta} = 2\,.$$

Hence, from monotonicity,

$$(Dc_{R,R+r_U})_{p\cdot\delta} = 2 \qquad \text{for all} \qquad p \in [-R, 0]\,.$$

Since δ is arbitrary, then

$$(Dc_{R,R+r_U})_{v} = 2 \qquad \text{for all} \qquad v \in B_{\tilde{R}}[0]\,. \tag{4.3}$$

Let us denote

$$\hat{R} = ((R + r_U)^2 - r_U^2)^{1/2}\,.$$

Notice, from (1.4), that

$$(Dc_{R,R+r_U})_{-\hat{R}\cdot\delta}$$

depends only on points of

$$B_{r_U}[-\hat{R}\cdot\delta] = \{\, v \in \mathbb{R}^2 \mid \|v + \hat{R}\cdot\delta\| \le r_U \,\}$$

as represented in Figure 4.3.

Consider the δ-configuration $\hat{y} : \mathbb{R}^2 \to M$ given by

$$\hat{y}(v) = \begin{cases} 1 & \text{if} \quad \langle v, \delta\rangle \ge -\hat{R}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice, vide Figure 4.3, that

$$c_{R,R+r_U}(v) \ge \hat{y}(v) \qquad \text{for all} \qquad v \in B_{r_U}[-\hat{R}\cdot\delta]\,.$$

Therefore, from monotonicity,

$$(Dc_{R,R+r_U})_{-\hat{R}\cdot\delta} \ge (D\hat{y})_{-\hat{R}\cdot\delta}\,. \tag{4.4}$$

Configuration $\hat{y}^{\delta} : \mathbb{R} \to M$ is given by

$$\hat{y}^{\delta}(p) = \begin{cases} 1 & \text{if} \quad p \ge -\hat{R}\,, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$(D^\delta \hat{y}^\delta)_p = (S^{V_{01}(D^\delta)}\hat{y}^\delta)_p = \begin{cases} 1 & \text{if} \quad p \geq -\hat{R} + V_{01}(D^\delta) \\ 0 & \text{otherwise.} \end{cases}$$

Since $V_{01}(D^\delta) \leq 0$,

$$(D^\delta \hat{y}^\delta)_{-\hat{R}} = 1\,.$$

From Lemma 20

$$(D\hat{y})_{-\hat{R}\cdot\delta} = (D^\delta \hat{y}^\delta)_{-\hat{R}} = 1\,. \tag{4.5}$$

Thus, from (4.4) and (4.5)

$$(Dc_{R,R+r_U})_{-\hat{R}\cdot\delta} \geq 1\,.$$

Hence, from monotonicity,

$$(Dc_{R,R+r_U})_{p\cdot\delta} \geq 1 \quad \text{for all} \quad p \in [-\hat{R}, -\tilde{R})\,.$$

Since δ is arbitrary, then

$$(Dc_{R,R+r_U})_v \geq 1 \quad \text{for all} \quad v \in B_{\hat{R}}[0] \cap (B_{\tilde{R}}[0])^c\,. \tag{4.6}$$

Notice that $\hat{R} > \tilde{R} + r_U$. Indeed, $\hat{R} > \tilde{R} + r_U \Leftrightarrow \hat{R}^2 > (\tilde{R} + r_U)^2 \Leftrightarrow (R + r_U)^2 - r_U^2 > \tilde{R}^2 + 2\tilde{R}r_U + r_U^2 \Leftrightarrow R^2 + 2Rr_U + r_U^2 - r_U^2 > \tilde{R}^2 + 2\tilde{R}r_U + r_U^2 \Leftrightarrow R^2 + 2Rr_U > R^2 - r_U^2 + 2\tilde{R}r_U + r_U^2$ $\Leftrightarrow R^2 + 2Rr_U > R^2 + 2\tilde{R}r_U \Leftrightarrow R > \tilde{R}$.

From (4.3) and (4.6) and since $\hat{R} > \tilde{R} + r_U$,

$$c_{\tilde{R},\tilde{R}+r_U} \prec Dc_{R,R+r_U}\,.$$

Lemma 27 is proved.

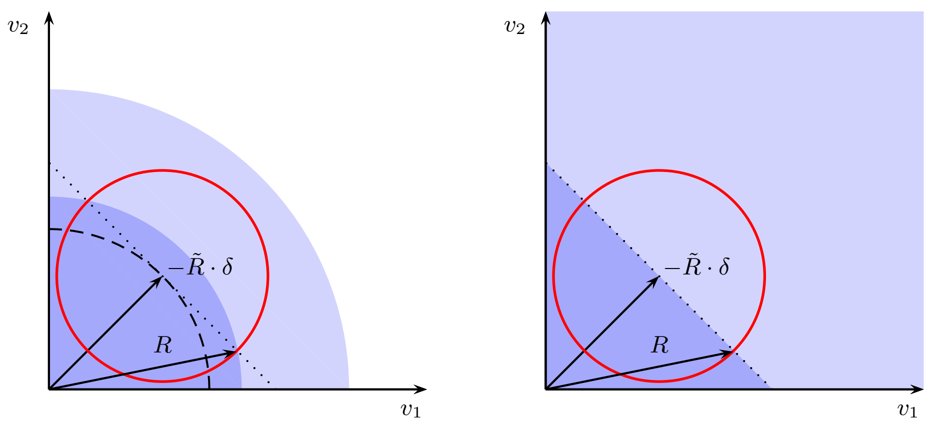

Figure 4.2: *Wedding cake $c_{R,R+r_U}$ and δ-configuration y.*

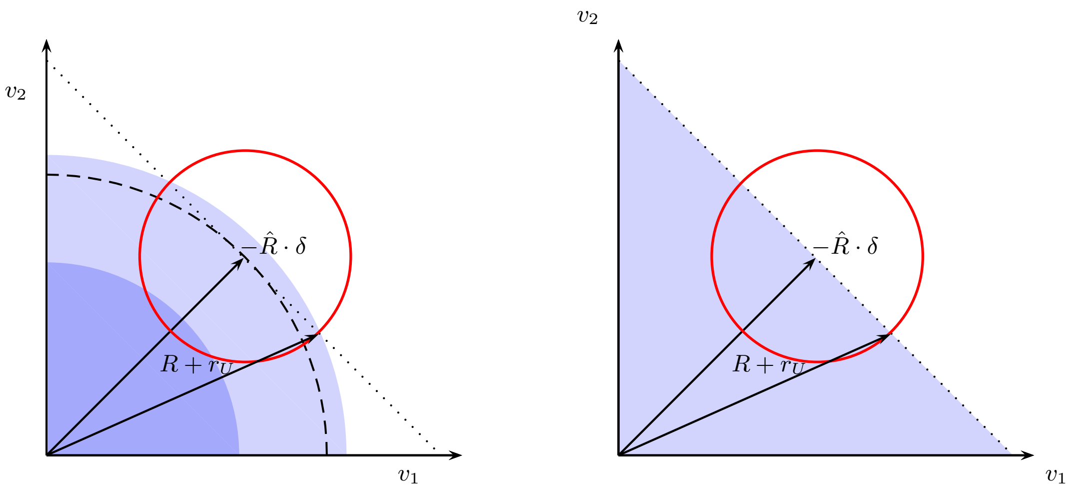

Figure 4.3: *Wedding cake* $c_{R,R+r_U}$ *and* δ*-configuration* $\hat{y}$*.*

Lemma 28 *Let* r_0 *be an element of* $[0, r_U^2)$*. Suppose that* $R_{02}(D^\delta) \le 0$ *and* $V_{01}(D^\delta) \le 0$ *for all direction* δ*. Then*

$$c_{r_0^{1/2}, r_0^{1/2}+r_U} \prec D^t c_{(t\,r_U^2+r_0)^{1/2},(t\,r_U^2+r_0)^{1/2}+r_U} \quad \textit{for all} \quad t \in \mathbb{Z}_+ .$$

Proof: From Lemma 27,

$$c_{(r_0)^{1/2},(r_0)^{1/2}+r_U} \prec D c_{(r_U^2+r_0)^{1/2},(r_U^2+r_0)^{1/2}+r_U} .$$

Thus it is true for $t = 1$.

Suppose that it is true for an arbitrary $t > 1$. From Lemma 27,

$$c_{(t\,r_U^2+r_0)^{1/2},(t\,r_U^2+r_0)^{1/2}+r_U} \prec D c_{((t+1)\,r_U^2+r_0)^{1/2},((t+1)\,r_U^2+r_0)^{1/2}+r_U} . \tag{4.7}$$

From monotonicity of D and (4.7),

$$D^t c_{(t\,r_U^2+r_0)^{1/2},(t\,r_U^2+r_0)^{1/2}+r_U} \prec D^{t+1} c_{((t+1)\,r_U^2+r_0)^{1/2},((t+1)\,r_U^2+r_0)^{1/2}+r_U} . \tag{4.8}$$

Thus, from (4.8) and the inductive hypothesis,

$$c_{r_0^{1/2},r_0^{1/2}+r_U} \prec D^t c_{(t\,r_U^2+r_0)^{1/2},(t\,r_U^2+r_0)^{1/2}+r_U} \prec D^{t+1} c_{((t+1)\,r_U^2+r_0)^{1/2},((t+1)\,r_U^2+r_0)^{1/2}+r_U} .$$

Thus it is true for $t + 1$.

Lemma 28 is proved.

Proof of Theorem 6: Let R be a real number such that $R > r_U$. Dividing R^2 by r_U^2, there is $t_0 \in \mathbb{Z}_+$ and $r_0 \in \mathbb{R}_+$ for which

$$0 \le r_0 < r_U^2 \tag{4.9}$$

and

$$R^2 = t_0\, r_U^2 + r_0\,. \tag{4.10}$$

From Lemma 28

$$c_{r_0^{1/2}, r_0^{1/2}+r_U} \prec D^{t_0} c_{R,R+r_U}\,.$$

Notice that $\tau_2^D(c_{r_0^{1/2}, r_0^{1/2}+r_U}) \geq 1$. Therefore

$$\tau_2^D(c_{R,R+r_U}) \geq t_0 + \tau_2^D(c_{r_0^{1/2}, r_0^{1/2}+r_U}) \geq t_0 + 1 \tag{4.11}$$

From (4.10),

$$t_0 + 1 = \frac{R^2}{r_U^2} - \frac{r_0}{r_U^2} + 1\,. \tag{4.12}$$

From (4.9),

$$1 - \frac{r_0}{r_U^2} > 0\,. \tag{4.13}$$

Thus from (4.11), (4.12) and (4.13)

$$\tau_2^D(c_{R,R+r_U}) > \frac{R^2}{r_U^2}\,.$$

Theorem 6 is proved.

4.2 Proof of Theorem 7

Lemma 29 *Suppose that there is $r > 0$ such that $-r > R_{02}(D^\delta)$ and $-r > V_{01}(D^\delta)$ for all direction δ. Let R be a positive real number such that*

$$R - (R^2 - r_U^2)^{1/2} < r\,.$$

Then

$$c_{\overline{R}, \overline{R}+2r_U} \prec D c_{R,R+2r_U}\,,$$

where $\overline{R} = r + (R^2 - r_U^2)^{1/2}$.

Proof: Notice that $\overline{R} > R$. Let us denote by $\tilde{R} = (R^2 - r_U^2)^{1/2}$ and $R' = ((R+2r_U)^2 - r_U^2)^{1/2}$. Consider the δ-configuration $y : \mathbb{R}^2 \to M$ given by

$$y(v) = \begin{cases} 2 & \text{if} \quad \langle v, \delta\rangle \geq -\tilde{R} \\ 1 & \text{if} \quad -R' \leq \langle v, \delta\rangle < -\tilde{R} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that

$$(Dc_{R,R+2r_U})_{p\cdot\delta} \geq (Dy)_{p\cdot\delta} \quad \text{for all} \quad p \in (-\infty, 0]\,. \tag{4.14}$$

Configuration $y^\delta : \mathbb{R} \to M$ is given by

$$y^\delta(p) = \begin{cases} 2 & \text{if} \quad p \geq -\tilde{R} \\ 1 & \text{if} \quad -R' \leq p < -\tilde{R} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $l(y^\delta) > 2r_U$. Hence, from Lemma 6,

$$(Dy^\delta)_p = \begin{cases} 2 & \text{if} \quad p \geq -\tilde{R} + V_{12}(D^\delta) \\ 1 & \text{if} \quad -R' + V_{01}(D^\delta) \leq p < -\tilde{R} + V_{12}(D^\delta) \\ 0 & \text{otherwise.} \end{cases}$$

From Lemma 20

$$(Dy)_{p\cdot\delta} = (D^\delta y^\delta)_p\,,$$

whence

$$(Dy)_{p\cdot\delta} = 2 \quad \text{for all} \quad p \in [-\tilde{R} + V_{12}(D^\delta), 0]\,,$$

and

$$(Dy)_{p\cdot\delta} \geq 1 \quad \text{for all} \quad p \in [-R' + V_{01}(D^\delta), -\tilde{R} + V_{12}(D^\delta))\,.$$

From (4.14)

$$(Dc_{R,R+2r_U})_{p\cdot\delta} = 2 \quad \text{for all} \quad p \in [-\tilde{R} + V_{12}(D^\delta), 0]\,,$$

and

$$(Dc_{R,R+2r_U})_{p\cdot\delta} \geq 1 \quad \text{for all} \quad p \in [-R' + V_{01}(D^\delta), -\tilde{R} + V_{12}(D^\delta))\,.$$

Notice that $\tilde{R} - V_{12}(D^\delta) \geq \tilde{R} + r = \overline{R}$. Observe also that $R' - V_{01}(D^\delta) \geq \overline{R} + 2r_U$. Since δ is arbitrary,

$$c_{\overline{R},\overline{R}+2r_U} \prec Dc_{R,R+2r_U}\,.$$

Lemma 29 is proved.

Lemma 30 *Consider the difference equation*

$$\begin{cases} R_t = r + (R_{t-1}^2 - r_U^2)^{1/2}, & t \in \mathbb{Z}_+\,, \\ R_0 = R\,, \end{cases}$$

where $R > r_U$ *and* $R - (R^2 - r_U^2) < r$*. Then*

$$R_t < R_{t+1} \quad \textit{for all} \quad t \in \mathbb{Z}_+ \tag{4.15}$$

and

$$\lim_{t \to \infty} R_t = \infty \,. \tag{4.16}$$

Proof: At first, let us prove (4.15). By hypothesis

$$R_0 = R < r + (R^2 - r_U^2)^{1/2} = R_1 \,.$$

So it is true for $t = 0$.

Suppose that it is true for an arbitrary $t > 0$. Now consider $g : (r_U, \infty) \to \mathbb{R}$ given by

$$g(z) = r + (z^2 - r_U^2)^{1/2} \,.$$

Notice that g is increasing. Indeed,

$$\frac{d}{dz} g(z) = \frac{z}{(z^2 - r_U^2)^{1/2}} > 0$$

From the inductive hypothesis, $R_t < R_{t+1}$. Then, since g is increasing,

$$R_{t+1} = g(R_t) < g(R_{t+1}) = R_{t+2} \,.$$

Thus it is true for $t + 1$.

Now let us show that

$$R_{t+1} - R_t < R_{t+2} - R_{t+1} \quad \text{for all} \quad t \in \mathbb{Z}_+ \,.$$

Consider the map $h : (r_U, \infty) \to \mathbb{R}$ given by

$$h(z) = r + (z^2 - r_U^2)^{1/2} - z \,.$$

Notice that h is also increasing. Indeed,

$$\frac{d}{dz} h(z) = \frac{z}{(z^2 - r_U^2)^{1/2}} - 1 > 0 \,.$$

Since

$$R_t < R_{t+1} \quad \text{for all} \quad t \in \mathbb{Z}_+ \,,$$

then, since h is increasing,

$$h(R_t) = R_{t+1} - R_t < R_{t+2} - R_{t+1} = h(R_{t+1}) \quad \text{for all} \quad t \in \mathbb{Z}_+ \,.$$

Thus

$$R_{t+1} = (R_{t+1} - R_t) + (R_t - R_{t-1}) + \cdots + (R_1 - R_0) + R_0 \geq t(R_1 - R) + R\,.$$

Therefore

$$\lim_{t\to\infty} R_t > \lim_{t\to\infty} t(R_1 - R) + R = \infty\,.$$

Lemma 30 is proved.

Lemma 31 *Suppose that there is $r > 0$ such that $-r \geq V_{01}(D^\delta)$ and $-r \geq R_{02}(D^\delta)$ for all direction δ. Then there is $R > 0$ such that*

$$c_{R_t,R_t+2r_U} \prec D^t c_{R,R+2r_U} \quad \textit{for all} \quad t \in \mathbb{Z}_+\,,$$

and $R_t = r + (R_{t-1}^2 - r_U^2)^{1/2}$ and $R_0 = R$.

Proof: From Lemma 29, it is true for $t = 1$.

Suppose that it is true for an arbitrary for $t > 1$. Notice, from (4.15), that

$$R_t - (R_t^2 - r_U^2) < r\,,$$

Then, from Lemma 29

$$c_{R_{t+1},R_{t+1}+2r_U} \prec D c_{R_t,R_t+2r_U}\,.$$

From the inductive hypothesis and monotonicity,

$$c_{R_{t+1},R_{t+1}+2r_U} \prec D c_{R_t,R_t+2r_U} \prec D^{t+1} c_{R,R+2r_U}\,.$$

Thus it is true for $t + 1$.

Lemma 31 is proved.

Proof of Theorem 7: Since $0 \in int(\sigma_D)$, then there is $\epsilon' > 0$ such that $0 > -\epsilon' \geq R_{02}(D^\delta)$ for all direction δ.

From Lemma 31, there is $R > 0$ such that

$$c_{R_t,R_t+2r_U} \prec D^t c_{R,R+2r_U} \quad \text{for all} \quad t \in \mathbb{Z}_+\,.$$

where $R_t = r + (R_{t-1}^2 - r_U^2)^{1/2}$ and $R_0 = R$. So, Theorem 7 follows from (4.16).

Theorem 7 is proved.

CHAPTER 5

Discussion and Perspectives

There remains always something to do; with sufficient study and penetration, we could improve any solution, and, in any case, we can always improve our understanding of the solution.

— **George Polya**

Someone familiar with Galperin (1975, 1977) and who has just read Chapter 2 may wonder why the author did not consider $M = \{0, 1, \ldots, m\}$. It was because the main purpose here was to deal with a natural next step after Toom's and Galperin's works, namely, the case $n = 2$ and $M = \{0, 1, 2\}$. Anyone who does mathematical research, even a very unexperienced one like the author, eventually realizes that many technical problems may appear from generalizations that at first sight seem to be pretty easy. Hence the author could spend so much time dealing with $M = \{0, 1, \ldots, m\}$ case that he should not work on his main goal.

5.1 Discrete space case

Although the continuous space case seemed to be a generalization of the discrete case, Lemma 32 clarifies how it works.

Lemma 32 *Let* $x : \mathbb{R}^n \to M$ *be an arbitrary* $\mathbb{R}^n$*-configuration. Consider* $\hat{x} : \mathbb{Z}^n \to M$ *such that*

$$x(v) = \hat{x}(v) \quad \textit{for all} \quad p \in \mathbb{Z}^n$$

(i.e., $\hat{x}$ is the restriction of map x to $\mathbb{Z}^n$). Let f be a transition map and U be a neighborhood such that its elements belong to $\mathbb{Z}^n$. Consider $P : M^{\mathbb{Z}^n} \to M^{\mathbb{Z}^n}$ and $D : M^{\mathbb{R}^n} \to M^{\mathbb{R}^n}$ defined by f and U according equations (1.1) and (1.4) respectively. Then

$$(D^t x)_v = (P^t \hat{x})_v \qquad \textit{for all} \qquad v \in \mathbb{Z}^n \quad \textit{and} \quad t \in \mathbb{Z}_+ .$$

Proof: Indeed it is true for $t = 0$ by hypothesis.
Suppose that it is true for an arbitrary natural $t > 0$, i.e.,

$$(D^t x)_p = (P^t \hat{x})_p \qquad \text{for all} \qquad p \in \mathbb{Z}^n .$$

Now, let p be an integer number. By definition

$$(D^{t+1} x)_p = f((D^t x)_{p+u_1}, \ldots, (D^t x)_{p+u_k}).$$

Since $p + u_1, \ldots, p + u_k \in \mathbb{Z}^n$, then by the inductive hypothesis

$$(D^t x)_{p+u_1} = (P^t \hat{x})_{p+u_1} \, , \; \ldots \; , \, (D^t x)_{p+u_k} = (P^t \hat{x})_{p+u_k} \, ,$$

and hence

$$(D^{t+1} x)_p = f((D^t x)_{p+u_1}, \ldots, (D^t x)_{p+u_k}) = f((P^t \hat{x})_{p+u_1}, \ldots, (P^t \hat{x})_{p+u_k}) = (P^{t+1} \hat{x})_p \, .$$

Thus it is true for $t + 1$.

Lemma 32 is proved.

Thus, once we have a continuous space result, its discrete space version becomes a Corollary. In particular, from Lemma 32 one can prove the discrete versions of Theorems 6 and 7. However, the author thinks that the proofs for Theorems 6 and 7 arose more natural to him than could arise if he tried at first to prove their discrete space versions.

It is another example of a very common situation in mathematics: some questions are easily answered when we make use of a larger mathematical structure.

5.2 Computation of velocities

In Chapter 2 we have proved, by two distinct ways, the existence of R_{02}. However,
Given a one-dimensional regular operator D, how can we actually compute R_{02}?
Until now we only know that a shift operator S^q has

$$R_{02} = L_{02} = q \, .$$

Before we answer that question, let us answer an easier question:

How can we compute V_{01}?

At first, if necessary, reorder U in such way that

$$u_1 < u_2 < \cdots < u_k \, .$$

Notice, from Lemma 5, that it is sufficient to solve the case where $u_1 = 0$. From Lemma 1, there is $V_{01} \in \mathbb{R}$ such that

$$Dj_{01} = S^{V_{01}} j_{01} \, .$$

Notice, from (1.4), that

$$(Dj_{01})_p = 0 \quad \text{for all} \quad p < -u_k \, .$$

But how about $(Dj_{01})_{-u_k}$? Well,

$$(Dj_{01})_{-u_k} = f(x_{-u_k+0}, x_{-u_k+u_2}, \ldots, x_{-u_k+u_{k-1}}, x_{-u_k+u_k}) = f(0,0,\ldots,0,1) \, .$$

If $f(0,0,\ldots,0,1) = 1$, then $V_{01} = -u_k$. Otherwise, from (1.4),

$$(Dj_{01})_p = 0 \quad \text{for all} \quad p < -u_{k-1} \, .$$

How about $(Dj_{01})_{-u_{k-1}}$?

$$(Dj_{01})_{-u_{k-1}} = f(x_{-u_{k-1}+0}, \ldots, x_{-u_{k-1}+u_{k-2}}, x_{-u_{k-1}+u_{k-1}}, x_{-u_{k-1}+u_k}) = f(0,\ldots,0,1,1) \, .$$

If $f(0,\ldots,0,1,1) = 1$, then $V_{01} = -u_{k-1}$. Otherwise, etc. Vide the following algorithm:

Algorithm 1 Compute velocity V_{01}.

Require: k, f, U ▷ function $f : M^k \to M$ and $U = \{0, u_2, \ldots, u_k\}$

$x \leftarrow (0,0,\ldots,0)$ ▷ $x \in M^k$
for $i \leftarrow 1, k$ **do**
    $x(k-i+1) \leftarrow 1$
    if $f(x) = 1$ **then**
        $V_{01} = -u_{k-i+1}$
        return V_{01}
    end if
end for

The algorithm for V_{12} is analogous.

If $V_{01} \leq V_{12}$, then, from Lemma 8,

$$L_{02} = V_{01} \qquad \text{and} \qquad R_{02} = V_{12} \, .$$

Thus the problem of computing R_{02} is solved when $V_{01} \leq V_{12}$.

However, how about the case where $V_{01} > V_{12}$?

Conjecture 2 *If $V_{01} > V_{12}$, then*

$$l(D^t j_{02}) = \sum_{i=1}^{k} \sum_{j=1}^{i} t_{i,j}(u_i - u_j)\,,$$

where $t_{i,j} \in \mathbb{Z}_+$ and

$$\sum_{i=1}^{k} \sum_{j=1}^{i} t_{i,j} = t\,.$$

Once one have proved Conjecture 2, the following conjecture is proved.

Conjecture 3 *The right (02)-velocity R_{02} can be computed in a finite number of operations.*

Indeed, If $V_{01} \leq V_{12}$, then it is done. Otherwise, there is x^* given by Lemma 13 such that

$$S^{l(x^*)} x^* \prec j_{02} \prec x^*\,.$$

From monotonicity

$$S^{l(x^*)} D^t x^* \prec D^t j_{02} \prec D^t x^*\,.$$

Therefore

$$0 \leq l(D^t j_{02}) \leq 2l(x^*) \quad \text{for all} \quad t \in \mathbb{Z}_+\,. \tag{5.1}$$

From Conjecture 2 and (5.1), the set $\{\, l(D^t j_{02}) \,|\, t \in \mathbb{Z}_+ \,\}$ is finite. Hence there is a $t_0 \in \mathbb{Z}_+$ such that the sequence

$$l(D^{t_0} j_{02}), l(D^{t_0+1} j_{02}), l(D^{t_0+2} j_{02}), \ldots$$

is periodic with period T. Therefore

$$R_{02} = \frac{\mathcal{R}(D^{t_0+T} j_{02}) - \mathcal{R}(D^{t_0} j_{02})}{T}\,.$$

5.3 When $\sigma_D \neq \emptyset$

There are two conjectures yet, namely,

Conjecture 4 *Let D be a two-dimensional regular operator. If $\sigma_D \neq \emptyset$, then there is a positive real number C such that*

$$\tau_2^D(d_R) \geq CR^2 \quad \textit{for all} \quad R \in \mathbb{R}_+\,.$$

Conjecture 5 *If $int(\sigma_D) \neq \emptyset$, then there is a positive real R such that*

$$\lim_{t\to\infty} diam\{\, v \in \mathbb{R}^2 \,|\, (D^t d_R)_v = 2\} = \infty\,.$$

At first, notice that Conjectures 4 and 5 are similar to Toom's conjecture and Conjecture 1 respectively. However, one can prove the following results:

Lemma 33 *If $\sigma_D \neq \emptyset$, then there is $q \in \mathbb{R}^2$ such that $\tilde{D} = S^q \circ D$ has $0 \in \sigma_{\tilde{D}}$.*

Lemma 34 *If $int(\sigma_D) \neq \emptyset$, then there is $q \in \mathbb{R}^2$ such that $\tilde{D} = S^q \circ D$ has $0 \in int(\sigma_{\tilde{D}})$.*

Therefore it is sufficient to prove (or disprove) Toom's conjecture and Conjecture 1. We have obtained favorable signs for them: Theorem 6 and Theorem 7. In order to prove (or disprove) these conjectures, the author guesses that it is better to be more acquainted on how $R_{02}(D^\delta)$, $V_{01}(D^\delta)$ and $V_{12}(D^\delta)$ change according to δ. Simulations should be useful for that purpose.

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"[...] materials alone are not enough for constructing a house but we cannot construct a house without collect the necessary materials."

— George Polya

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